POISSON SUMMATION FORMULA

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The Poisson summation formula (PSF) is a mathematical technique that relates the sum of a function over the integers to its Fourier transform. It allows one to transfer information between the continuous and discrete domains and has important applications in fields such as signal processing, complex analysis and number theory. The formula was first discovered by the French mathematician Siméon Poisson [1] in the early 19th century and is sometimes called Poisson resummation. To introduce and prove the PSF we introduce the following background information and preliminaries.

1. Background Information

1.1. **n-Torus $\mathbb{T}^n$.** The n-torus or n-dimensional torus extends the idea of a torus to higher dimensions. The common torus is a two-dimensional surface that can be created by gluing the opposing corners of a rectangular piece of paper to create a tube. The resulting surface has a hole in the middle and can be visualized as a doughnut or a bagel. The n-torus $\mathbb{T}^n$ is the cube $[0,1]^n$ with opposite sides identified. Specifically, we consider two points in the cube to be identified if they differ by an integer in every coordinate. Now we note that functions on $\mathbb{T}^n$ are functions f on $\mathbb{R}^n$ that satisfy $f(x+m)=f(x)$ for all $x \in \mathbb{R}^n$ and $m \in \mathbb{Z}^n$. Such functions are called 1-*periodic* in every coordinate. This is true because of the way the torus is constructed - by identifying points in $\mathbb{R}^n$ that differ by an integer vector. As a result, these functions repeat themselves every time we translate by an integer vector in any direction.

1.2. **Lebesgue Space $L_p(\mathbb{R}^n)$.** The L_p spaces are used to measure the size of a function. Intuitively, the L_p spaces provide a was to measure how spreadout a function is, with different values of p corresponding to different ways of measuring spread. For instance, the L_1 space is used to measure the absolute spread of a function, the L_2 space is used to measure the squared spread of a function.

Definition 1.1. For $0<p<\infty$, $L_p(\cdot)$ is defined as the space of all f defined on $\mathbb{R}^n$, such that $\|f\|_{L_p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{1/p}$ and $\|f\|_{L_p(\mathbb{T}^n)} := \left(\int_{\mathbb{T}^n} |f(x)|^p dx\right)^{1/p}$ are finite.

Note. For any $f \in L_1(\mathbb{T}^n)$, then $\int_{\mathbb{T}^n} f(t)dt = \int_{[0,1]^n} f(t)dt = \int_{[-1/2,1/2]^n} f(t)dt$.

1.3. **Fourier Series and Transform.** The Fourier series is a way of representing a periodic function as a sum of sine and cosine functions with different amplitudes and frequencies. This enables us to express a large range of functions in terms of simple trigonometric functions. On the other hand, the Fourier Transform is a powerful tool that helps us convert a function $f(x)$ defined in the (usually time) $x \in \mathbb{R}$ domain to another function $\widehat{f}(\xi)$ in the (frequency) ξ-domain which describes the frequency spectrum of the function f which is typically represented as a graph showing the amplitude or magnitude of each frequency component as a function of frequency. The Fourier series and Fourier transform are related in that the Fourier transform can be thought of as the extension of the Fourier series to non-periodic signals. Moreover, a periodic signal's Fourier series coefficients can be calculated using the Fourier transform, and vice versa. These have numerous applications in signal processing, harmonic analysis, and differential equations.

Definition 1.2. A trigonometric series $f_s(t)$ on $\mathbb{T}^n$ is an expression of the form: $f_s(t) = \sum_{k\in\mathbb{Z}^n} c_k e^{ik\cdot t}, t \in \mathbb{R}^n, c_k \in \mathbb{C}$.

Definition 1.3. If the set $\{k \in \mathbb{Z}^n : c_k \neq 0\}$ is finite, then it is called a trigonometric polynomial and $f(t) = \sum_{k\in\mathbb{Z}^n} c_k e^{ik\cdot t}$. In view of the orthonormality of the exponentials we have $\forall k \in \mathbb{Z}^n,\ c_k = \int_{\mathbb{T}^n} f(t) e^{-ik\cdot t} dt,\ k \in \mathbb{Z}^n \iff c_k = \widehat{f}(k)$ (1.5).

Definition 1.4. The Fourier series of a function $f \in L_1(\mathbb{T}^n)$ is defined as $\sum_{k\in\mathbb{Z}^n} \widehat{f}(k) e^{2\pi i k\cdot t}, t \in \mathbb{R}^n$ where the k th Fourier coefficient $\widehat{f}(k)$ of f is $\widehat{f}(k) = \int_{\mathbb{T}^n} f(t) e^{-2\pi i k\cdot t} dt, k \in \mathbb{Z}^n$.

Definition 1.5. The Fourier transform of a function $f \in L_1(\mathbb{R}^n)$ is defined as $\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi} dx, \xi \in \mathbb{R}^n$

Definition 1.6. The inverse Fourier transform of a function $f \in L_1(\mathbb{R}^n)$ is defined as $\check{f}(x) := \int_{\mathbb{R}^n} f(\xi) e^{2\pi i \xi\cdot x} d\xi, x \in \mathbb{R}^n$

1.4. **Fejér Kernel.** The Fejér Kernel is a family of functions that are used in mathematical analysis and signal processing to approximate other functions. It is named after the Hungarian mathematician Lipót Fejér [2], who introduced it in the early 20th century. In the context of Fourier analysis, a kernel is often used to smooth or regularize a function. The Fejér kernel is called a *kernel* because it is usually convoluted with a function to produce a smoothed version of that function. The Fejér kernel is used to approximate a function $f(x)$ on the torus by taking a linear combination of f convoluted with the Fejér kernel. Here we define convolution for two functions to be $(f * g)(x) := \frac{1}{2\pi} \int_{\mathbb{R}^n} f(x-t) g(t) dt$.

Definition 1.7. The Fejér Kernel $(F_{\tilde{n}}(t))$ is defined as $F_{\tilde{n}}(t) = \sum_{|k|\leq \tilde{n}-1} \left(1 - \frac{|k|}{\tilde{n}}\right) e^{ikt}$ and in n dimensions it is defined to be $F_{\tilde{n}}^n(t_1,\ldots,t_n) := F_{\tilde{n}}(t_1)\cdots F_{\tilde{n}}(t_n)$. We notice that it is a trigonometric polynomial as it is a finite sum.

Now that we have stated all the necessary background information we now state the theorem and give a proof for it.

2. Poisson summation formula

The motivation for the PSF comes from the Fourier series, which expresses a periodic function as a sum of sinusoidal functions. The Fourier transform is a generalization of the Fourier series to non-periodic functions. The PSF relates the Fourier transform of a function on the real line to its values on the integers. It states that if a function and its Fourier transform satisfy certain conditions, then the sum of the values of the function on the integers is equal to the sum of the values of its Fourier transforms on the integers. This result is remarkable because it relates a continuous function on the real line to a discrete sequence of values on the integers. The PSF provides a way to calculate the values of a function at integer points using its Fourier transform, which may be easier to compute or manipulate.

Theorem 2.1. (PSF) A.5.7 in Han[3], 3.2.8 in Grafakos[4] *Suppose that* $f, \widehat{f} \in L_1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ *satisfy* $|f(x)| \leq C(1+|x|)^{-n-\varepsilon}$, $\forall x \in \mathbb{R}^n$, *for some* $C, \varepsilon > 0$ *and whose Fourier transform* $\widehat{f}$ *restricted on* $\mathbb{Z}^n$ *satisfies* $\sum_{m\in\mathbb{Z}^n} |\widehat{f}(m)| < \infty$. *Then we have the relation,*

$$\sum_{k\in\mathbb{Z}^n} \widehat{f}(k)e^{2\pi ik\cdot x} = \sum_{k\in\mathbb{Z}^n} f(x+k),\ \forall x \in \mathbb{R}^n$$

Before we give a proof of the PSF we have to state some preliminaries.

3. Preliminaries

Theorem 3.1. (Convolution Theorem) A.3.1 in [3], 3.1.2 in [4] *If* $f \in L_p(\mathbb{T}^n)$ *with* $1 \leq p \leq \infty$ *and* $g \in L_1(\mathbb{T}^n)$, *then* $[f * g](x)$ *is well defined for a.e* $x \in \mathbb{R}^n, f * g \in L_p(\mathbb{T}^n)$ *and* $\widehat{f * g}(k) = \widehat{f}(k)\widehat{g}(k)$ *for all* $k \in \mathbb{Z}^n$.

Proof: We have

$$\int_{[-1/2,1/2]^n}\int_{[-1/2,1/2]^n} |f(t-s)g(s)|dtds = \int_{[-1/2,1/2]^n} |f(t)|dt \int_{[-1/2,1/2]^n} |g(s)|ds = \|f\|_1\|g\|_1 \leq \|f\|_p\|g\|_1 < \infty$$

Since $f \in L_p(\mathbb{T}^n)$ and $g \in L_1(\mathbb{T})$, we conclude that $f(t-s)g(s) \in L_1\left([-1/2,1/2]^{2n}\right)$. By this fact and by Fubini's Theorem we can interchange the order of integration,

$$\widehat{f * g}(k) = \int_{[-1/2,1/2]^n}\int_{[-1/2,1/2]^n} f(t-s)g(s)e^{-2\pi ik\cdot t}dsdt = \int_{[-1/2,1/2]^n}\int_{[-1/2,1/2]^n} f(t-s)e^{-2\pi ik\cdot(t-s)}g(s)e^{-2\pi ik\cdot s}dtds$$

$$= \int_{[-1/2,1/2]^n} f(t)e^{-2\pi ik\cdot t}dt\left(\int_{[-1/2,1/2]^n} g(s)e^{-2\pi ik\cdot s}ds\right) = \widehat{f}(k)\widehat{g}(k),\ \forall k \in \mathbb{Z}^n. \qquad \square$$

Note. Using [3.1] on a function $f \in L_1(\mathbb{R}^n)$ and $F_{\tilde{n}(t)}$ gives us $f * F_{\tilde{n}}(t) = \sum_{j=1-\tilde{n}}^{\tilde{n}-1}\left(1 - \frac{|j|}{\tilde{n}}\right)\widehat{f}(j)e^{ijt}$.

Theorem 3.2. (Fejér) A.3.2 in [3], 1.2.19 in [4] *If* $f \in L_p(\mathbb{T}^n)$ *with* $1 \leq p \leq \infty$ *then* $\lim_{\tilde{n}\to\infty} \|f * F_{\tilde{n}}^n - f\|_p = 0$.

As we don't provide a proof for the above theorem we just give a brief note about the theorem. It says that for $f \in L_p(\mathbb{T}^n)$ with $1 \leq p \leq \infty$, the sequence of partial Fourier sums of f converges to f in the L_p norm as the number of terms in the sum goes to infinity. The theorem ensures that the approximation error for say approximating a signal can be made arbitrarily small by choosing a sufficiently large number of terms in the partial Fourier sum.

Using the above results we get the following corollaries and propositions:

Corollary. (Weierstrass approximation theorem) A.3.2 in [3], 1.2.19 in [4] *The set of trigonometric polynomials is dense in* $L_p(\mathbb{T}^n)$ *for* $1 \leq p < \infty$.

Proof: Given f in $L_p(\mathbb{T}^n)$ for $1 \leq p < \infty$, consider $f * F_{\tilde{n}}^n$. From the above note, $f * F_{\tilde{n}}^n$ is also a trigonometric polynomial. From (3.2]) , $f * F_{\tilde{n}}^n$ converges to f in L_p as $\tilde{n} \to \infty$ $\square$.

The Weierstrass approximation theorem is motivated by the need to approximate continuous functions with simpler functions like polynomial functions that are easier to work with. This has various uses in numerical methods like polynomial interpolation, numerical integration and areas such as functional analysis and signal and image processing.

Corollary. A.3.4 in [3], 3.2.4 in [4] *Let* $f, g \in L_1(\mathbb{T}^n)$. *If* $\widehat{f}(k) = \widehat{g}(k)$ *for all* $k \in \mathbb{Z}^n$, *then* $f(t) = g(t)$ *a.e.* $t \in \mathbb{R}^n$.

Proof: Let $h = f - g$. Then $\widehat{h}(k) = 0$ for all $k \in \mathbb{Z}^n$. Therefore, $h * F_{\tilde{n}}^n = 0$ for all $\tilde{n} \in \mathbb{N}$. By (3.2]) , $h = \lim_{\tilde{n}\to\infty} h * F_{\tilde{n}}^n = 0$ in $L_1(\mathbb{T}^n)$. Hence, $h = 0$ and $f = g$ a.e. $p < \infty$. $\square$.

Proposition 3.3. (Fourier inversion) 3.2.5 in [4] *Suppose that* $f \in L_1(\mathbb{T}^n)$ *and that* $\sum_{m\in\mathbb{Z}^n} |\widehat{f}(m)| < \infty$. *Then* $f(x) = \sum_{m\in\mathbb{Z}^n} \widehat{f}(m)e^{2\pi im\cdot x}$ *a.e., and therefore* f *is almost everywhere equal to a continuous function.*

Proof: By definition, both functions are well-defined and have the same fourier coefficients. Therefore, they must be almost everywhere equal by the above corollary. Also the function of the right is continuous everywhere. $\square$

Remark. If f, $\widehat{f}$ *are in* $L_1(\mathbb{R}^n)$, *then* $f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)e^{2\pi i\xi\cdot x}d\xi$, *a.e.* $x \in \mathbb{R}^n$. *Therefore, both* $f, \widehat{f} \in L_1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$.

The Fourier Inversion formula shows that the original function can be recovered from its Fourier transform by the inverse Fourier transform formula. Now, using the above results we are ready to provide a detailed proof of the PSF.

4. Proof of PSF

Proof: We follow the proof 3.2.8 in [4] but add and justify any necessary details. Define a function $f^{\text{per}}(\cdot)$ on $\mathbb{T}^n$, $f^{\text{per}}(x) = \sum_{k\in\mathbb{Z}^n} f(x+k)$. This is clearly 1- periodic as $f^{\text{per}}(x+1) = \sum_{k\in\mathbb{Z}^n} f(x+k+1) = \sum_{\tilde{k}\in\mathbb{Z}^n} f(x+\tilde{k}) = f^{\text{per}}(x)$ where $\tilde{k} = k+1$. By the assumption that $|f(x)| \le C(1+|x|)^{-n-\varepsilon}$, $\forall x \in \mathbb{R}^n$, the above series converges absolutely and uniformly. Hence, $f^{\text{per}} \in C_0(\mathbb{T}^n) \subset L_1(\mathbb{T}^n)$. In fact we justify that $\|f^{\text{per}}\|_{L^1([0,1]^n)} = \|f\|_{L^1(\mathbb{R}^n)}$.

$$\|f^{\text{per}}\|_{L_1([0,1]^n)} = \int_{\mathbb{T}^n}\left|\sum_{k\in\mathbb{Z}^n} f(x+k)\right| dx \le \int_{\mathbb{T}^n}\sum_{k\in\mathbb{Z}^n}|f(x+k)|\,dx = \sum_{k\in\mathbb{Z}^n}\int_{\mathbb{T}^n}|f(x+k)|\,dx = \sum_{k\in\mathbb{Z}^n}\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^n-k}|f(x)|$$

which is equal to $\int_{\mathbb{R}^n}|f(x)|dx = \|f\|_{L^1(\mathbb{R}^n)}$. The other direction is similar by using the roles of f^{per} and f reversed. Now we prove that the sequence of the Fourier coefficients of f^{per} equals to the restriction of the Fourier transform of f on $\mathbb{Z}^n$. This follows from :

$$\widehat{f^{\text{per}}}(m) = \int_{\mathbb{T}^n}\sum_{k\in\mathbb{Z}^n} f(x+k)e^{-2\pi i m\cdot x}dx = \sum_{k\in\mathbb{Z}^n}\int_{\mathbb{T}^n} f(x+k)e^{-2\pi i m\cdot x}dx = \sum_{k\in\mathbb{Z}^n}\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^n-k} f(x)e^{-2\pi i m\cdot x}dx$$

which is equal to $\int_{\mathbb{R}^n} f(x)e^{-2\pi i m\cdot x}dx = \widehat{f}(m)$ where $x \leftarrow x+k$. Now we justify the interchange of sum and integral in the above proofs. For any $x \in \mathbb{T}^n = [0,1]^n$, we note that $0 \le |x| \le \sqrt{n}$ as $|x| := (|x_1|^2 + \cdots + |x_n|^2)^{1/2}$ for $x \in \mathbb{T}^n$. Therefore, by $|f(x)| \le C(1+|x|)^{-n-\varepsilon}$, $\forall x \in \mathbb{R}^n$, and by the Weierstrass M-test of uniform convergence of series,

$$f^{\text{per}}(x) = \sum_{k\in\mathbb{Z}^n} f(x+k) \le \sum_{k\in\mathbb{Z}^n}\frac{1}{(1+|k+x|)^{n+\varepsilon}} \le \sum_{k\in\mathbb{Z}^n}\frac{(1+\sqrt{n})^{n+\varepsilon}}{(1+\sqrt{n}+|k+x|)^{n+\varepsilon}} \le \sum_{k\in\mathbb{Z}^n}\frac{C_{n,\varepsilon}}{(1+|k|)^{n+\varepsilon}} < \infty,$$

where we used $|k+x| \ge |k|-|x| \ge |k|-\sqrt{n}$. This calculation also shows that f^{per} is the sum of a uniformly convergent series of continuous functions on $[0,1]^n$, thus it is itself continuous. Hence, Prop (3.3) applies, and given the fact that f^{per} is continuous, it yields,

$$\sum_{m\in\mathbb{Z}^n}\widehat{f}(m)e^{2\pi i m\cdot x} = \sum_{k\in\mathbb{Z}^n} f(x+k)$$

for all $x \in \mathbb{T}^n$ and, by periodicity, this holds for all $x \in \mathbb{R}^n$. □

5. Applications

We now mention some application of PSF:

- In number theory, PSF can be used to derive a variety of functional equations including the functional equation for the Riemann zeta function. It is also a fundamental tool in the study of the distribution of integers and prime numbers.
- In Signal processing, PSF can be used in digital signal processing to create filters and examine a signal's frequency content. Moreover, it can be utilised to apply Fourier transforms to discrete data.
- In Differential equations, by converting them into a sum over the Fourier coefficients of the solution, PSF can be used to solve some differential equations (PDEs). It can also be used to examine how partial differential equation solutions behave near singular points. It can be used to obtain explicit solutions to certain PDEs.
- In Computer science, PSF can be used in computer graphics and image processing to perform Fourier transforms on discrete data. It can also be used to design algorithms for certain optimization problems.
- In Combinatorics, PSF is used in combinatorial number theory to count the number of lattice points in certain regions. It can also be used to obtain asymptotic formulas for certain combinatorial sums.
- In Probability Theory, PSF can be used to compute the characteristic function of a random variable and to derive the Fourier transform of a probability density function and in the study of random processes.

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