Linear Independence and Stability of Integer Shifts of Functions

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Abstract

This poster is about the linear independence and stability of functions. I begin by discussing preliminary issues before moving on to the linear independence of functions and a few of the theorems and results we proved during the research process. Then I give a brief overview of stability and its applications, followed by a discussion on some limitations I encountered.

Preliminaries

A Distribution is a *generalized function* that extends the notion of a function beyond the classical definition of a function as a rule that assigns a unique output to each input. Instead, a distribution assigns a value to a *test function*, which is a smooth function with compact support.

A distribution (on $\mathbb{R}^d$) is a continuous linear functional (**dual**) $f : \mathscr{D}(\mathbb{R}^d) \to \mathbb{C}$, i.e.,

(i) It is linear

$$f(c_1\varphi_1 + c_2\varphi_2) = c_1 f(\varphi_1) + c_2 f(\varphi_2)$$

for all $c_1, c_2 \in \mathbb{C}$ and all $\varphi_1, \varphi_2 \in \mathscr{D}$,

(ii) and continuous in the sense that

$$f(\varphi_k) \to f(\varphi) \quad \text{as} \quad \varphi_k \to \varphi \text{ (in } \mathscr{D}).$$

Denote the space of all distributions by $\mathscr{D}'(\mathbb{R}^d)$ and we denote $f(\varphi) := \langle f, \varphi \rangle$.

The Schwartz space is the space of rapidly decreasing functions on $\mathbb{R}^d$. Similar to how $\mathscr{D}'(\mathbb{R}^d)$ is the dual to $\mathscr{D}(\mathbb{R}^d)$ we have the Tempered distribution space denoted by $\mathscr{S}'(\mathbb{R}^d)$ being the dual of the Schwartz function space $\mathscr{S}(\mathbb{R}^d)$.

The space $\mathscr{E}'(\mathbb{R}^d)$ of distributions with compact support on $\mathbb{R}^d$ is the linear space of all distributions in $\mathscr{D}'\mathbb{R}^d)$ with compact support.

The Fourier Transform (Figure 1) is a tool that helps us convert a function $f(x)$ defined in the (usually time) $x \in \mathbb{R}$ domain to another function $\widehat{f}(\xi)$ in the (frequency) ξ- domain. For $f \in \mathscr{S}(\mathbb{R}^d)$ is

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx$$

and if $f \in \mathscr{S}'(\mathbb{R}^d)$, then

$$\langle \widehat{f}, \varphi \rangle := \langle f, \widehat{\varphi} \rangle, \ \varphi \in \mathscr{S}(\mathbb{R}^d)$$

The Fourier Transform of the integer shift of a function $\widehat{\varphi}(\cdot)$ is $\widehat{\varphi}(\cdot - k) = e^{-2\pi i \xi \cdot k}\widehat{\varphi}(\xi)$ for $k \in \mathbb{R}^d$ and similarly, for every $f \in \mathscr{S}'(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $k \in \mathbb{R}^d$,

$$\langle \widehat{f}(x-k), \varphi(x) \rangle = \langle f(x), \widehat{\varphi}(x+k) \rangle$$

The Dirac Delta (Figure 2) and its integer shift are defined as

$$\langle \delta_k, \varphi \rangle := \varphi(k), \left\langle \widehat{\delta_k}, \varphi \right\rangle = \left\langle e^{-2\pi i \xi \cdot k}, \varphi \right\rangle, \quad \delta_k = \delta(\cdot - k)$$

Poisson Summation Formula

The Poisson Summation Formula (PSF) is a very remarkable theorem in the field of Classical Fourier Analysis which relates a continuous function on the real line to a discrete sequence of values on the integers using the Fourier transform. Given some preliminary conditions for a function $f \in \mathscr{S}(\mathbb{R}^d)$ and its Fourier transform i.e both have to be rapidly decreasing the PSF states,

$$\sum_{k \in \mathbb{Z}^d} \widehat{f}(k) e^{2\pi i k \cdot x} = \sum_{k \in \mathbb{Z}^d} f(x+k), \ \forall x \in \mathbb{R}^d$$

Linear Independence

For $v = \{v_k\}_{k \in \mathbb{Z}^d} \in \ell(\mathbb{Z}^d)$, we say $\{f(\cdot - k)\}_{k \in \mathbb{Z}^d}$ is called $\ell(\mathbb{Z}^d)$ - linearly independent if $\sum_{k \in \mathbb{Z}^d} v_k f(\cdot - k) = 0 \implies v_k = 0 \ \forall \ k \in \mathbb{Z}^d$ where $\ell(\mathbb{Z}^d)$ is the space of all sequences. Linear independence is important for integer shifts of compact support functions because it allows for stability and shift-invariance i.e, when the functions are linearly independent, the shifted versions are also linearly independent, which means that the shifted functions retain their unique characteristics and can be distinguished from one another. This property is handy in signal processing and image processing.

We show one example of linear independence of vectors in $\mathbb{R}^3$ in Figure 3 and they are linearly independent because they are neither collinear nor coplanar.

Proven Results

Claim 1: Let $(c_k)_{k \in \mathbb{Z}^d} \in \ell(\mathbb{Z}^d)$ such that $|c_k| \le A(1 + |k|)^M$ for all k and some fixed M and $A > 0$. Let δ_k denote Dirac mass at the integer k. Then the sequence $\sum_{|k| \le n} c_k \delta_k$ converges to some tempered distribution f in $\mathscr{S}'(\mathbb{R}^d)$ as $n \to \infty$. Also $\widehat{f}$ is the $\mathscr{S}'$ limit of the sequence of functions $f_n(\xi) = \sum_{|k| \le n} c_k e^{-2\pi i \xi \cdot k}$.

Remarks: Why do we need the above claim?

(i) It help us define the infinite sums $f(x) = \sum_{k \in \mathbb{Z}^d} c_k \delta_k$ and $\widehat{f}(\xi) = \sum_{|k \in \mathbb{Z}^d} c_k e^{-2\pi i \xi \cdot k}$ in $\mathscr{S}'$.

(ii) Also while proving this result one shows that taking Fourier transform of $f(x) = \sum_{|k| \le n} c_k \delta_k$ is justifiable.

The following lemma essentially proves Linear Independence for the functions $\{e^{-2\pi \cdot k}\}_{k \in \mathbb{Z}^d}$.

Lemma: Let $\widehat{f} = \sum_{k \in \mathbb{Z}^d} c_k e^{-2\pi i \xi \cdot k} \in \mathscr{E}'(\mathbb{R}^d)$. If $\widehat{f} = 0$ then $c_k = 0 \quad \forall k \in \mathbb{Z}^d$ where $(c_k)_{k \in \mathbb{Z}^d} \in \ell(\mathbb{Z}^d)$ such that $|c_k| \le A(1 + |k|)^M$ for all k and some fixed M and $A > 0$.

The following Theorem proves Linear Independence for integer shifts of functions and is a direct consequence of the above lemma and by the definition of the Fourier Transform.

Theorem A: Let $0 \not\equiv f \in \mathscr{E}'(\mathbb{R}^d)$. Then for $x \in \mathbb{R}^d, \sum_{k \in \mathbb{Z}^d} c_k f(x - k) = 0$ implies $c_k = 0$ where $(c_k)_{k \in \mathbb{Z}^d} \in \ell(\mathbb{Z}^d)$ such that $|c_k| \le A(1 + |k|)^M$ for all k and some fixed M and $A > 0$.

Theorem B: Let $f \in \mathscr{E}'(\mathbb{R}^d)$ and consider the sets $V = \{c \in \ell(\mathbb{Z}) : c * f = 0\}$ and $S = \{\xi \in \mathbb{C}^d. : \widehat{f}(\xi + 2\pi k) = 0, \forall k \in \mathbb{Z}^d\}$. We then have $\{e^{i\xi \cdot k}\}_{k \in \mathbb{Z}^d} \in V$ if and only if $\xi \in S$ for some $\xi \in \mathbb{C}^d$. where $c * f$ is actually formally called the Discrete Convolution defined as: $c * f = \sum_{k \in \mathbb{Z}^d} c_k f(\cdot - k)$. The Discrete Convolution is essentially a mathematical operation that basically allows one to shift and multiply one sequence by the other, and then sum the resulting values.

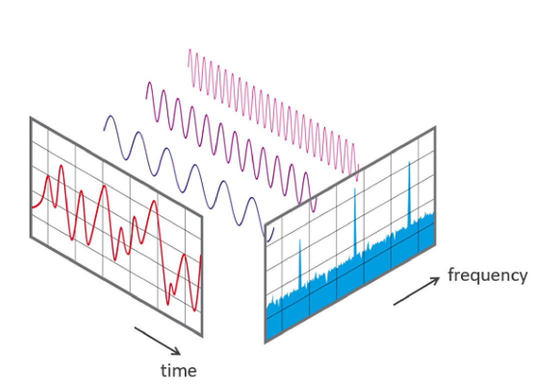

Figure 1. Fourier Transform

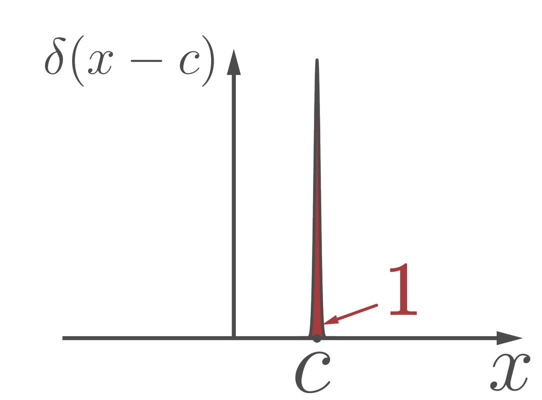

Figure 2. Shifted Dirac Delta

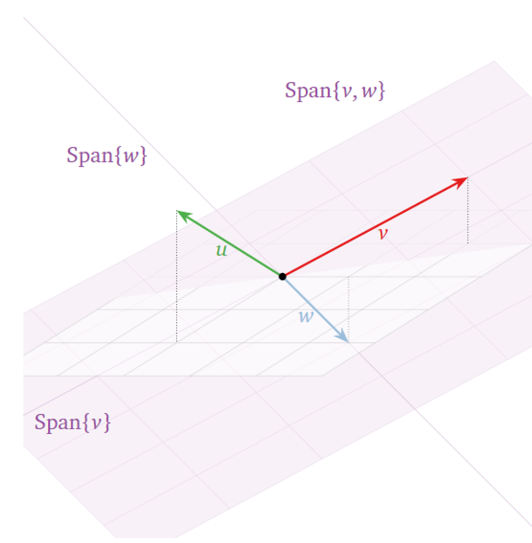

Figure 3. Linear Independence

Stability

The concept of stability plays an important role in approximation theory and wavelet analysis. Stability is an important concept in the study of integer shifts of functions because it ensures that small perturbations in the input function do not result in large changes in the shifted output function.

For $1 \le p \le \infty$, we say that the integer shift of a compactly supported distribution $f \in L_p(\mathbb{R}^d)$ is stable in $L_p(\mathbb{R}^d)$ if there exist positive constants C_1 and C_2 such that for all $\{v_k\}_{k \in \mathbb{Z}^d} \in \ell_p$ we have,

$$C_1 \|v_k\|_{\ell_p(\mathbb{Z}^d)} \le \left\| \sum_{k \in \mathbb{Z}^d} v_k f(\cdot - k) \right\|_{L_p(\mathbb{R}^d)} \le C_2 \|v_k\|_{\ell_p(\mathbb{Z}^d)}$$

Note: The proof for showing the stability of integer shifts of functions in $L_p(\mathbb{R})$ is done via characterization which states that if f is a compactly supported function defined on $\mathbb{R}^d$ and let $\widehat{f}$ be it's Fourier transform then, the integer shift of f is stable if and only if $\widehat{f}$ does not possess in $\mathbb{R}^d$ any $2\pi k$-periodic zeros, i.e., the set $S' = \{\xi \in \mathbb{R}^d : \widehat{f}(\xi + 2\pi k) = 0, \ \forall k \in \mathbb{Z}\}$ is empty. Comparing the set S in Theorem B with the above set S' we get that linear independence implies stability i.e there are examples of functions that have stability but are not linearly independent.

Applications

Linearly independent integer shifts of a function are important for interpolation because they form a set of basis functions that can be used to represent any function that is band-limited to the same cut-off frequency.

Linearly independent and stability of integer shifts are important concepts in the numerical solution of partial differential equations. These concepts are closely related to the concept of numerical stability, which is essential for obtaining accurate and reliable solutions to partial differential equations. For example, if a solution is stable under integer shifts, then it can be discretized effectively on a grid with a finite number of points.

Stability is also used in the reconstruction of signals known as the Nyquist-Shannon Sampling Theorem.

Limitations

Although stability is one of the fundamental concepts for my project, the proofs and text regarding stability were very involved and required background material from areas of math that I was not familiar with. This presented a barrier to me as I was not familiar with these prerequisite concepts and spent most of my time learning the background material regarding Distribution Theory and Fourier analysis to get a good grasp of these concepts.

References

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