Linear Independence and Stability of Integer Shifts of Functions

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Introduction

Harmonic Analysis: Branch of mathematics concerned with investigating the connections between a function and its representation in frequency.

- Study the spread of waves in fluids and solids.
- Study of the **Fourier series** and **Fourier transforms**.
- Used in Signal and Image Processing.

Fourier Analysis $\subset$ Harmonic Analysis

Let $\varphi : \mathbb{R}^d \to \mathbb{C}$ be a function. We say that φ is a test function if:

1. φ is compactly supported i.e there exists a compact (closed an bounded) set $K \subset \mathbb{R}^d$ such that $\varphi(x) = 0$ for all $x \notin K$.

2. φ is C^∞.

A distribution (on $\mathbb{R}^d$) is a continuous linear functional (**dual**) $f : \mathscr{D}\left(\mathbb{R}^d\right) \to \mathbb{C}$, i.e.,

(i) It is linear

$$f\left(c_1\varphi_1 + c_2\varphi_2\right) = c_1 f\left(\varphi_1\right) + c_2 f\left(\varphi_2\right)$$

for all $c_1, c_2 \in \mathbb{C}$ and all $\varphi_1, \varphi_2 \in \mathscr{D}$,

(ii) and continuous in the sense that

$$f\left(\varphi_k\right) \to f(\varphi) \quad \text{as} \quad \varphi_k \to \varphi \text{ (in } \mathscr{D}).$$

Denote the space of all distributions by $\mathscr{D}'\left(\mathbb{R}^d\right)$ and we denote $f(\varphi) := \langle f, \varphi \rangle$.

The Schwartz space or space of rapidly decreasing functions on $\mathbb{R}^d$ is the function space

$$\mathscr{S}(\mathbb{R}^d) := \{\varphi \in C^\infty(\mathbb{R}^d) \mid \forall \alpha, \beta \in \mathbb{N}_0^d,\ \rho_{\alpha,\beta} < \infty\},$$

where

$$\rho_{\alpha,\beta} := \sup_{x \in \mathbb{R}^d} \left| x^\alpha \partial^\beta \varphi(x) \right| = \left\| x^\alpha \partial^\beta \varphi(x) \right\|_\infty$$

For simplicity we let $\beta = (0, 0, \cdots 0) \in \mathbb{N}_0^d$.

Distribution Theory Cont...

Tempered Distributions

Similar to how $\mathscr{D}'(\mathbb{R}^d)$ is the dual to $\mathscr{D}(\mathbb{R}^d)$ we have the Tempered distribution space denoted by $\mathscr{S}'(\mathbb{R}^d)$ being the dual of the Schwartz function space $\mathscr{S}(\mathbb{R}^d)$.

The space $\mathscr{E}'\left(\mathbb{R}^d\right)$ of distributions with compact support on $\mathbb{R}^d$ is the linear space of all distributions in $\mathscr{D}'\left(\mathbb{R}^d\right)$ with compact support.

A linear functional f on $\mathscr{S}\left(\mathbb{R}^d\right)$ is a tempered distribution if and only if there exist $C > 0$ and k, m integers such that

$$|\langle f, \varphi \rangle| \leq C \sum_{|\alpha| \leq m, |\beta| \leq k} \rho_{\alpha,\beta}(\varphi), \quad \forall \varphi \in \mathscr{S}\left(\mathbb{R}^d\right)$$

Distribution Theory Cont...

How do these spaces relate to each other?

The dual spaces are nested as follows:

$$\mathscr{E}'\left(\mathbb{R}^d\right) \subseteq \mathscr{S}'\left(\mathbb{R}^d\right) \subseteq \mathscr{D}'\left(\mathbb{R}^d\right)$$

In fact we have that $\mathscr{S}\left(\mathbb{R}^d\right) \subseteq L_p\left(\mathbb{R}^d\right) \subseteq \mathscr{S}'\left(\mathbb{R}^d\right)$ for $p \in [1, \infty]$.

We *sketch* the idea behind the proof of the second inequality.

Proof.
First - For every $\varphi \in \mathscr{S}\left(\mathbb{R}^d\right)$, we have

$$\left(\left|\left(1 + x^{\tilde{n}}\right)\varphi(x)\right|\right)^p \leq \left\|\left(1 + x^{\tilde{n}}\right)\varphi\right\|_\infty^p = \left\|\varphi + x^{\tilde{n}}\varphi\right\|_\infty^p$$

$$\leq (\rho_{0,0}(\varphi) + \rho_{\tilde{n},0}(\varphi))^p < \infty$$

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Second - Consider $\int_{\mathbb{R}^d} f(x)\varphi(x)dx$,

$$\left|\int_{\mathbb{R}^d} f(x)\varphi(x)dx\right| \leq \int_{\mathbb{R}^d} |f(x)||\varphi(x)|\frac{1+|x|^{\tilde{n}}}{1+|x|^{\tilde{n}}}dx \leq$$

$$(\rho_{0,0}(\varphi)+\rho_{\tilde{n},0}(\varphi))^p \int_{\mathbb{R}^d} \frac{|f(x)|}{1+|x|^{\tilde{n}}}dx$$

Third - Let $C = (\rho_{0,0}(\varphi) + \rho_{\tilde{n},0}(\varphi))^p$, and using the fact that the function $1/\left(1 + |x|^{\tilde{n}}\right)$ is in L_p, we have by Holder's Inequality that,

$$\left| \int_{\mathbb{R}^d} f(x)\varphi(x)dx \right| \leq C \int_{\mathbb{R}^d} \frac{|f(x)|}{1 + |x|^{\tilde{n}}} dt \leq C \|f\|_p \left\| \frac{1}{1 + |x|^{\tilde{n}}} \right\|_{\tilde{p}} < \infty$$

where $\tilde{p}$ is such that $1/p + 1/\tilde{p} = 1$

Therefore $\int_{\mathbb{R}^d} f(x)\varphi(x)dx$ is defined and let this be $f(\varphi) = \langle f, \varphi \rangle = \int_{\mathbb{R}^d} f(x)\varphi(x)dx$ and by the check we mentioned before $f(\varphi)$ is a tempered distribution. $\square$

Given $f \in \mathscr{S}\left(\mathbb{R}^d\right)$ the Fourier transform $(\widehat{f}(\xi))$ is

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx$$

and if $f \in \mathscr{S}'\left(\mathbb{R}^d\right)$, then

$$\langle \widehat{f}, \varphi \rangle := \langle f, \widehat{\varphi} \rangle, \ \varphi \in \mathscr{S}(\mathbb{R}^d)$$

Actually, if $\varphi \in \mathscr{S}\left(\mathbb{R}^d\right), f \in \mathscr{S}'\left(\mathbb{R}^d\right)$ then $\widehat{\varphi} \in \mathscr{S}\left(\mathbb{R}^d\right), \widehat{f} \in \mathscr{S}'\left(\mathbb{R}^d\right)$

Fourier Transforms Cont...

What is $\widehat{\varphi}(x-k)$?

Since our project deals with integer shifts of functions, the following is an obvious consequence of the above definitions,

If $\varphi \in \mathscr{S}\left(\mathbb{R}^d\right)$ and $k \in \mathbb{R}^d$, then $\widehat{\varphi}(x-k) = e^{-2\pi i \xi \cdot k}\widehat{\varphi}(\xi)$

Proof. Define $\tilde{\varphi} := \varphi(x-k)$. Then we have

$$\widehat{\tilde{\varphi}}(x) = \int_{\mathbb{R}^d} \tilde{\varphi}(x) e^{-2\pi i x.\xi} dx = \int_{\mathbb{R}^d} \varphi(x-k) e^{-2\pi i x.\xi} dx$$

$$= e^{-2\pi i \xi.k} \int_{\mathbb{R}^d} \varphi(x-k) e^{-2\pi (x-k).\xi} dx = e^{-2\pi i \xi.k} \widehat{\varphi}(\xi)$$

with the change of variable $x \leftarrow x-k$ $\square$.

Fourier Transforms Cont...
The dirac delta

Similarly, for every $f \in \mathscr{S}'(\mathbb{R}^d), x \in \mathbb{R}^d$ and $k \in \mathbb{R}^d$,

$$\langle \widehat{f}(x-k), \varphi(x) \rangle = \langle f(x), \widehat{\varphi}(x+k) \rangle$$

and

$$\langle \delta_k, \varphi \rangle := \varphi\left(k\right), \left\langle \widehat{\delta_k}, \varphi \right\rangle = \left\langle e^{-2\pi i \xi \cdot k}, \varphi \right\rangle, \quad \delta_k = \delta(\cdot - k)$$

Suppose that $f, \widehat{f} \in L_1\left(\mathbb{R}^d\right) \cap C\left(\mathbb{R}^d\right)$ *and* $f \in \mathscr{S}(\mathbb{R}^d)$ *i.e* $|f(x)| \leq C(1+|x|)^{-n-\varepsilon},\ \forall x \in \mathbb{R}^d$, *for some* $C, \varepsilon > 0$ *and whose Fourier transform* $\widehat{f}$ *restricted on* $\mathbb{Z}^d$ *satisfies* $\sum_{m\in\mathbb{Z}^d} |\widehat{f}(m)| < \infty$. *Then we have the relation,*

$$\sum_{k\in\mathbb{Z}^d} \widehat{f}(k)e^{2\pi i k\cdot x} = \sum_{k\in\mathbb{Z}^d} f(x+k),\ \forall x \in \mathbb{R}^d$$

Linear Independence

For $v = \{v_k\}_{k\in\mathbb{Z}^d} \in \ell(\mathbb{Z}^d)$, we say $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}^d}$ is called $\ell(\mathbb{Z}^d)$ - linearly independent if $\sum_{k\in\mathbb{Z}^d} v_k\phi(\cdot - k) = 0 \implies v_k = 0 \;\forall\; k \in \mathbb{Z}^d$.

Linear Independence Cont..

Let $(c_k)_{k\in\mathbb{Z}^d} \in \ell\left(\mathbb{Z}^d\right)$ such that $|c_k| \leq A(1+|k|)^M$ for all k and some fixed M and $A>0$. Let δ_k denote Dirac mass at the integer k. Then the sequence $\sum_{|k|\leq n} c_k\delta_k$ converges to some tempered distribution f in $\mathscr{S}'\left(\mathbb{R}^d\right)$ as $n\to\infty$. Also $\widehat{f}$ is the $\mathscr{S}'$ limit of the sequence of functions $f_n(\xi) = \sum_{|k|\leq n} c_k e^{-2\pi i\xi\cdot k}$.

Linear Independence Cont..

Why do we need the above claim?

(i) It help us define the infinite sums $f(x) = \sum_{k \in \mathbb{Z}^d} c_k \delta_k$ and $\widehat{f}(\xi) = \sum_{|k \in \mathbb{Z}^d} c_k e^{-2\pi i \xi \cdot k}$ in $\mathscr{S}'$.

(ii) Also while proving this result one shows that taking Fourier transform of $f(x) = \sum_{|k| \leq n} c_k \delta_k$ is justifiable.

Linear Independence Cont...

Now we finally come to the main part of the presentation:

(i) Let $\widehat{f} = \sum_{k \in \mathbb{Z}^d} c_k e^{-2\pi i \xi \cdot k} \in \mathscr{E}' \left(\mathbb{R}^d\right)$. If $\widehat{f} = 0$ then $c_k = 0 \quad \forall k \in \mathbb{Z}^d$ where $(c_k)_{k \in \mathbb{Z}^d} \in \ell\left(\mathbb{Z}^d\right)$ such that $|c_k| \leq A(1+|k|)^M$ for all k and some fixed M and $A > 0$.

(ii) Let $0 \not\equiv f \in \mathscr{E}' \left(\mathbb{R}^d\right)$. Then for $x \in \mathbb{R}^d, \sum_{k \in \mathbb{Z}^d} c_k f(x-k) = 0$ implies $c_k = 0$ where $(c_k)_{k \in \mathbb{Z}^d} \in \ell\left(\mathbb{Z}^d\right)$ such that $|c_k| \leq A(1+|k|)^M$ for all k and some fixed M and $A > 0$.

Linear Independence Cont...

Sketch of Proofs.

Proof. (i). We show this in parts:

I - Show the finite sum $f_n(\xi) := \sum_{1\leq k\leq n} c_k e^{-2\pi i\xi\cdot k}$ is linear independent using the Wronskian.

Recall: Let f, g be differentiable on some interval I. If there is one point $x_0 \in I$ such that

$$\det\begin{pmatrix} f\left(x_0\right) & g\left(x_0\right) \\ f'\left(x_0\right) & g'\left(x_0\right) \end{pmatrix} \neq 0,$$

then f, g are linearly independent.

II- Following a similar calculation above we also get that $\sum_{|k|\leq n} c_k e^{-2\pi i\xi\cdot k} = 0 \Longrightarrow c_k = 0$. Now from (R.1) we know that the infinite sum $\sum_{k\in\mathbb{Z}^d} c_k e^{-2\pi i\xi\cdot k}$ is defined and using the fact that $\mathscr{E}' \subseteq \mathscr{S}'$. Since we showed that for arbitrary finite n, $f_n(\xi)$ is linearly independent, we can conclude that the infinite sum $\widehat{f}$ is linearly independent. $\square$

Linear Independence Cont...

Sketch of Proofs.

Proof. (ii). For some $x \in \mathbb{R}^d$ consider $\sum_{k\in\mathbb{Z}^d} c_k f(x-k) = 0$. Taking Fourier Transform on both sides gives us,

$$\sum_{k\in\mathbb{Z}^d} c_k f(x-k) = 0 \implies \sum_{k\in\mathbb{Z}^d} c_k e^{-2\pi i \xi\cdot k} \widehat{f}(\xi) = 0$$

$$\implies \widehat{f}(\xi) \sum_{k\in\mathbb{Z}^d} c_k e^{-2\pi i \xi\cdot k} = 0$$

Linear Independence Cont...

Sketch of Proofs.

Now $\widehat{f}(\xi) \not\equiv 0$ as $f \not\equiv 0$. Therefore we must have $\sum_{k \in \mathbb{Z}^d} c_k e^{-2\pi i \xi \cdot k} = 0$.
By (i) we get that $\sum_{k \in \mathbb{Z}^d} c_k f(x-k) = 0$ implies $c_k = 0$. $\square$

Linear Independence Cont...

Sketch of Proofs.

Let $f \in \mathscr{E}'\left(\mathbb{R}^d\right)$ and consider the sets $V = \{v \in \ell(\mathbb{Z}) : v * f = 0\}$ and $S = \left\{\xi \in \mathbb{C}^d \,:\, \widehat{f}(\xi + 2\pi k) = 0, \forall k \in \mathbb{Z}^d\right\}$. We then have $\left\{e^{i\xi\cdot k}\right\}_{k\in\mathbb{Z}^d} \in V$ if and only if $\xi \in S$ for some $\xi \in \mathbb{C}^d$.

Here $v * f$ is the semi- discrete convolution defined as:

$$v * f = \sum_{k\in\mathbb{Z}^d} v_k f(\cdot - k)$$

Proof. ($\Longleftarrow$) Consider the set S as defined above. If $\exists \xi \in \mathbb{C}^d \ni \xi \in S$ then $\widehat{f}(\xi + 2\pi k) = 0, \forall k \in \mathbb{Z}^d$. Now by PSF for some $x \in \mathbb{R}^d$, we get

$$\sum_{k\in\mathbb{Z}} \widehat{f}(\xi + 2\pi k)e^{i2\pi x\cdot k} = \sum_{k\in\mathbb{Z}} f(x-k)e^{-i\xi(x-k)} = \sum_{k\in\mathbb{Z}} f(x-k)e^{-i\xi\cdot x}e^{i\xi\cdot k}$$

We have $\widehat{f}(\xi + 2\pi k) = 0, \forall k \in \mathbb{Z}^d$. Thus

$$\sum_{k\in\mathbb{Z}} f(x-k)e^{-i\xi\cdot x}e^{i\xi\cdot k} = 0 \Longrightarrow e^{-i\xi\cdot x}\sum_{k\in\mathbb{Z}} f(x-k)e^{i\xi\cdot k} = 0$$

Since, $e^{-i\xi\cdot x} \neq 0 \Longrightarrow \sum_{k\in\mathbb{Z}} f(x-k)e^{i\xi\cdot k} = 0$. Which is essentially $e^{i\xi k} * f = 0$. Take $v(k) = e^{i\xi\cdot k}$.

Linear Independence Cont...

Sketch of Proofs.

($\Longrightarrow$) Let $v(k) = e^{i\xi \cdot k} \in V$. Therefore we get $e^{i\xi \cdot k} * f = 0$. Multiply both sides by $e^{i\xi \cdot x}$. Using PSF and going in the reverse direction similar to the above direction by taking $c_k = \widehat{f}(\xi + 2\pi k)$ and using the above results we get the claim.

Stability

For $1 \leq p \leq \infty$, we say that the integer shift of a compactly supported distribution $f \in L_p(\mathbb{R}^d)$ is stable in $L_p(\mathbb{R}^d)$ if there exist positive constants C_1 and C_2 such that for all $\{v_k\}_{k\in\mathbb{Z}^d} \in \ell_p$ we have,

$$C_1\|v_k\|_{\ell_p(\mathbb{Z}^d)} \leq \left\|\sum_{k\in\mathbb{Z}^d} v_k f(\cdot - k)\right\|_{L_p(\mathbb{R}^d)} \leq C_2\|v_k\|_{\ell_p(\mathbb{Z}^d)}$$

Stability Cont...

Stability is an important concept in the study of integer shifts of functions because it ensures that small perturbations in the input function do not result in large changes in the shifted output function.

Stability Cont...

The proof for showing the stability of integer shifts of functions in $L_p(\mathbb{R})$ is done via characterization which states that if f is a compactly supported function defined on $\mathbb{R}^d$ and let $\widehat{f}$ be it's Fourier transform then, the integer shift of f is stable if and only if $\widehat{f}$ does not possess in $\mathbb{R}^d$ any $2\pi k$-periodic zeros, i.e., the set $S' = \left\{\xi \in \mathbb{R}^d : \hat{f}(\xi + 2\pi k) = 0,\ \forall k \in \mathbb{Z}\right\}$ is empty.

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Thank you