LINEAR INDEPENDENCE AND STABILITY OF INTEGER SHIFTS OF FUNCTIONS

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Abstract. This thesis explores the linear independence and stability of generalized functions also known as Distributions. We start by investigating the properties of Distributions and their relation to the Lebesgue spaces, the Fourier Transform and using this we prove some results for the Linear Independence of the Distributions. We later briefly mention about the stability of Distributions and provide applications and talk about some limitations encountered during the research process.

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1. Acknowledgments

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2. Introduction

Harmonic Analysis is the branch of mathematics concerned with investigating the connections between a function and its representation in frequency. It is used in Wavelet analysis, Signal and Image Processing and in the study of the Fourier Transform, Fourier Series, Shift Invariant Spaces, Refinable functions and Stability of Functions. Classical Fourier Analysis and Modern Fourier Analysis are $\subset$ of Harmonic Analysis and the former studies the Fourier Transform, Series and Convolution. On the other hand, the latter deals with Hardy and Besov Spaces, some Kernels and Multilinear Harmonic Analysis and could be considered as a continuation of the former.

Our project deals with the Linear Independence and Stability of Compactly Supported Distributions which is a topic in Harmonic Analysis and is studied in-depth from the Classical Fourier Analysis perspective. Our research goals essentially are to provide proofs to show the Linear independence of these distributions, relate the Kernel of the space of semi-discrete convolutions of a distribution to the Fourier transform of the distribution and provide some correlation between Stability and Linear Independence.

We start by providing some introduction to Distribution Theory and its relation to our favorite familiar Lebesgue Space which is mentioned in Grafakos [4]. Later on we talk about the Fourier Transform and one important result in Fourier Analysis - The Poisson Summation Formula which is usually found in any text relating to Fourier Analysis. Then we explicitly state the results of showing that compactly supported distributions are Linearly Independent which is mentioned by Han in [3], Christensen in [7] and also how the Kernel of the space of semi-discrete convolutions of distribution relates to the Linear Independence of Fourier transform of the distribution. Both the above topics are thoroughly studied by Han in [3] and Ros in [8]. To demonstrate the proof of the results, we initially acquainted ourselves with the relevant topics and concepts. We then studied the proofs presented in the reference materials, which provided us with some insight into how to approach the problem and, crucially, how to structure our own proofs. Finally, we used our understanding of fundamental mathematical concepts and tricks, which would be familiar to a competent undergraduate mathematics student i.e haven taken some upper level rigorous mathematic courses, to prove the results. We then briefly talk about Stability and one particular case of Stability i.e in the Hilbert Space. Sequences which satisfy the stability condition in the Hilbert Space are called Riesz Sequences and this is again thoroughly studied in Christensen [7], Han [3] and Jia [9]. As the proofs and explanations related to Stability were considerably more involved, and we lacked the necessary mathematical background to fully grasp them, we were not able to delve deeply into these topics.

We have only provided proofs for the theorems, lemmas, propositions, and remarks that we were able to prove on our own, with reference to familiar mathematical concepts. For those theorems and other mathematical results that we could not prove ourselves, either because they were too complex or we couldn't prove from start to finish on our own, or because they were straightforward consequences of established results, we have not included their proofs. However, we have mentioned these results as they served as the building blocks for our own results.

The proof of Linear Independence is fundamental to the study of Shift Invariant Subspaces and Refinable Vector Functions, which are extensively researched in the works of Han [3] and Ros [8]. This property is particularly important in signal processing and image processing, where shift-invariance enables us to differentiate between various functions. When functions are linearly independent, their shifted versions also maintain their unique characteristics, allowing them to be distinguished from one another. On the other hand, Stability is widely used in approximation theory and wavelet analysis because it ensures that small perturbations in the input function do not result in large changes in the shifted output function. One of the key usages of stability is in reconstruction of signals in signal processing using the sinc function.

Lastly we talk about the limitations we encountered during the research process.

3. Distribution Theory

A Distribution is a *generalized function* that extends the notion of a function beyond the classical definition of a function as a rule that assigns a unique output to each input. Instead, a distribution assigns a value to a *test function*, which is a smooth function with compact support (A.1).

The theory of distributions was introduced by Laurent Schwartz in the 1940s. A key benefit of using distributions in mathematical analysis is that they possess well-defined derivatives that are also distributions, and hence have infinitely many derivatives in the distributional sense. This property makes distributions a powerful tool for studying functions that may not have well-defined derivatives in the classical sense, and allows for a more general approach to solving differential equations and other mathematical problems.

Distributions can be defined on any open set $U \subseteq \mathbb{R}^d$. However we just consider the general case $(\mathbb{R}^d)$ as there mostly no changes. We denote the space of **test** functions as $\mathscr{D}(\mathbb{R}^d)$. A test function belongs to $\mathscr{C}^\infty(\mathbb{R}^d)$ and has compact support. The space $\mathscr{D}\left(\mathbb{R}^d\right)$ is endowed with the topology that $f_k \to f(\text{in } \mathscr{D})$, as $k \to \infty$, if all the f_k are supported in the same compact set K, and for any multi-index $\alpha \in \mathbb{N}^d$, $\lim_{k\to\infty} \|\partial^\alpha (f_k - f)\|_\infty = 0$, where $\|\cdot\|_\infty$ stands for the **supremum norm**. Now we come to the definition of a distribution:

Definition 3.1. Distributions: The space $\mathscr{D}'(\mathbb{R}^d)$ of distributions on $\mathbb{R}^d$ is the dual (i.e a continuous linear functional) of $\mathscr{D}(\mathbb{R}^d)$. This means a mapping $f : \mathscr{D}\left(\mathbb{R}^d\right) \to \mathbb{C}$ is called a distribution if it is a linear functional (A.3) and if $\varphi_k \to \varphi$ as $k \to \infty$ in $\mathscr{D}(\mathbb{R}^d)$, then $\lim_{k\to\infty} f(\varphi_k) = f(\varphi)$. We say that $f_k \to f$ in $\mathscr{D}'(\mathbb{R}^d)$ as $k \to \infty$ if all $f, f_k \in \mathscr{D}'(\mathbb{R}^d)$ and $\lim_{k\to\infty} f(\varphi_k) = f(\varphi)$ for all $\varphi \in \mathscr{D}(\mathbb{R}^d)$.

If $\varphi \in \mathscr{D}\left(\mathbb{R}^d\right)$, we will write its image with $f \in \mathscr{D}'\left(\mathbb{R}^d\right)$ as $\langle f, \varphi\rangle := f(\varphi)$. We do this as for a distribution f, the notation $\langle f, \varphi\rangle$, where $\varphi \in \mathscr{D}(\mathbb{R}^d)$, is understood that f acts on test functions from the space $\mathscr{D}$. Here $\langle f, \varphi\rangle$ is the inner product.

Theorem 3.1. A.6.1 in [3] *A linear functional $f : \mathscr{D}(\mathbb{R}^d) \to \mathbb{C}$ is a distribution on Ω (that is, $f \in \mathscr{D}'(\mathbb{R}^d)$) if and only if for every compact set $K \subset \mathbb{R}^d$, there exist a constant $C > 0$ and an integer $m \in \mathbb{N}_0$ such that*

$$|\langle f, \varphi\rangle| \le C \sum_{|\beta|_1 \le m} \left\|\partial^\beta \varphi\right\|_\infty, \quad \forall \varphi \in \mathscr{D}(\mathbb{R}^d) \text{ with support inside } K.$$

The following result loosens the continuity condition if we know a functional is linear.

Proposition 3.2. *Consider a linear functional $f : \mathscr{D}\left(\mathbb{R}^d\right) \to \mathbb{C}$ then if $\varphi_m \to 0$ in $\mathscr{D}\left(\mathbb{R}^d\right)$, then $\langle f, \varphi_m\rangle \to 0$ in $\mathbb{C}$ in order for f to be continuous, and thus a distribution.*

Proof. Let $\varphi_k \to \varphi$ be a convergent sequence in $\mathscr{D}\left(\mathbb{R}^d\right)$ then we get $\{\varphi_k - \varphi\} \to 0$ in $\mathscr{D}$. Then by definition $f, \langle f, \varphi_k - \varphi\rangle \to 0$. Because of linearity, $\langle f, \varphi_k - \varphi\rangle = \langle f, \varphi_k\rangle - \langle f, \varphi\rangle$, and we get that $\langle f, \varphi_k\rangle \to \langle f, \varphi\rangle$ □.

Example 3.1. Let φ be a testing function in $\mathbb{R}^d$. The functional $\delta : \mathscr{D}\left(\mathbb{R}^d\right) \to \mathbb{C}$ given by $\langle \delta, \varphi\rangle = \varphi(0)$ is called Dirac delta function which is a distribution. It is clear that the delta function is linear; $\langle \delta, \alpha\varphi + \tilde{\alpha}\tilde{\varphi}\rangle = \alpha\varphi(0) + \tilde{\alpha}\tilde{\varphi}(0) = \alpha\langle \delta, \varphi\rangle + \tilde{\alpha}\langle \delta, \tilde{\varphi}\rangle$, for every $\varphi, \tilde{\varphi} \in \mathscr{D}$ and $\alpha, \tilde{\alpha} \in \mathbb{C}$. For continuity, let $\varphi_n \to \varphi(0)$ in $\mathcal{D}$. Now $\langle \delta, \varphi_n\rangle = \varphi_n(0)$ and we need to show $\langle \delta, \varphi_n\rangle \to \langle \delta, \varphi\rangle$ in $\mathbb{C}$. We have uniform convergence in $\mathscr{D}$, therefore $\lim_{n\to\infty} \|\varphi_n - \varphi(0)\|_\infty = 0$. Therefore, $\lim_{n\to\infty} |\varphi_n(0 - \varphi(0)| = 0$.

3.1. Schwartz Functions.

The Schwartz space, denoted by $\mathscr{S}(\mathbb{R}^d)$, is a space of smooth functions on $\mathbb{R}^d$ that decay rapidly at infinity and have all their derivatives also decay rapidly at infinity. It is a fundamental class of test functions used in the theory of distributions and Fourier analysis. The Schwartz space is named after the French mathematician Laurent Schwartz who introduced it in the 1940s as a way to provide a rigorous foundation for the Fourier transform.

Definition 3.2. The Schwartz class: $\mathscr{S}(\mathbb{R}^d)$ consists of all $C^\infty(\mathbb{R}^d)$ functions f such that for every $\alpha, \beta \in \mathbb{N}_0^d$, $\rho_{\alpha,\beta}(f) := \left\|x^\alpha \partial^\beta f(x)\right\|_\infty < C_{\alpha,\beta} < \infty$. The quantities $\rho_{\alpha,\beta}(f)$ are called the Schwartz seminorms of f .

The following alternate characterization of the Schwartz class is more intuitive.

Note. A $\mathscr{C}^\infty$ function f is in $\mathscr{S}$ if and only if for all positive integers N and all multi-indices α there exists a positive constant $C_{\alpha,N}$ such that $|(\partial^\alpha f)(x)| \leq C_{\alpha,N}(1+|x|)^{-N}$. Intuitively, this means that the function and all its derivatives approach zero faster than any inverse power of the distance from the origin as x tends to infinity. Schwartz functions are important because they form a natural class of test functions for distributions.

Therefore we have that

$$\mathscr{D}\left(\mathbb{R}^d\right) \subseteq \mathscr{S}\left(\mathbb{R}^d\right) \subseteq C^\infty\left(\mathbb{R}^d\right)$$

Example 3.2. Consider $f(x) = e^{-x^2}$, to see that $f(x)$ belongs to the Schwartz space $\mathscr{S}(\mathbb{R})$, we need to check that $f(x)$ is smooth and that its derivatives decay rapidly as $|x| \to \infty$. The function $f(x)$ is clearly infinitely differentiable, and its nth derivative can be expressed as $f^{(n)}(x) = (-1)^n e^{-x^2} \sum_{k=0}^{n} \binom{n}{k}(2x)^{n-k}(2k-1)!!$, let $P_n(x) = \sum_{k=0}^{n} \binom{n}{k}(2x)^{n-k}(2k-1)!!$ and $P_n(x)$ denotes the polynomial of degree n that can be expressed in terms of the *Hermite* polynomials.

It can be shown that $|P_n(x)| \leq C_n(1+x^2)^{n/2}$ for all $x \in \mathbb{R}$, where C_n is a constant that depends on n. This implies that $|f^{(n)}(x)| \leq C_n e^{-x^2}(1+x^2)^{n/2}$ for all $x \in \mathbb{R}$, which shows that the derivatives of $f(x)$ decay rapidly as $|x| \to \infty$. Therefore, $f(x)$ belongs to the Schwartz space $\mathscr{S}(\mathbb{R})$.

However, $f(x)$ is not compactly supported, since it does not vanish outside any bounded interval. In fact, $f(x)$ is a positive function that is nonzero for all $x \in \mathbb{R}$, and its support is the entire real line. Therefore, $f(x)$ is an example of a function that belongs to the Schwartz space but is not compactly supported. □

We say that for f_k, f in $\mathscr{S}\left(\mathbb{R}^d\right)$ $\forall k \in \mathbb{N}$, f_k converges to f in $\mathscr{S}\left(\mathbb{R}^d\right)$ if for all multi-indices α and β we have $\rho_{\alpha,\beta}(f_k - f) = \left\|x^\alpha\left(\partial^\beta(f_k - f)\right)(x)\right\|_\infty \to 0$ as $k \to \infty$. In fact, 3.2.8 in [4] correlates convergence in $\mathscr{S}$ and the L_p space (A.4), it says if $f, f_k, \in \mathscr{S}\left(\mathbb{R}^d\right)$ $\forall k \in \mathbb{N}$. If $f_k \to f$ in $\mathscr{S}$ then $f_k \to f$ in L_p $\forall p \in (0,\infty]$.

The following is another result found in Page 119 in [4] which we will prove and will be useful in the results that follow.

Theorem 3.3. *$\mathscr{S}\left(\mathbb{R}^d\right) \subseteq L_p\left(\mathbb{R}^d\right)$ for $p \in [1,\infty]$.*

Proof. Let $f \in \mathscr{S}(\mathbb{R}^d)$, $\beta = (0,0,\cdots 0) \in \mathbb{N}_0^d$,we need to show $f \in L_p\left(\mathbb{R}^d\right)$. Consider,

$$\int_{\mathbb{R}^d} |f(x)|^p dx = \int_{\mathbb{R}^d} \frac{\left((1+|x|^{\tilde{n}})\,|f(x)|\right)^p}{(1+|x|^{\tilde{n}})^p} dx = \int_{\mathbb{R}^d} \left(\left|(1+|x|^{\tilde{n}})\,f(x)\right|\right)^p \frac{1}{(1+|x|^{\tilde{n}})^p} dx$$

Now $\left(\left|\,(1+|x|^{\tilde{n}})\,f(x)\right|\right)^p \leq \sup_{x\in\mathbb{R}^d}(|\,(1+x^{\tilde{n}})f(x)|)^p = \left\|(1+x^{\tilde{n}})\,f\right\|_\infty^p = \left\|f + x^{\tilde{n}} f\right\|_\infty^p \leq (\rho_{0,0}(f) + \rho_{\tilde{n},0}(f))^p < \infty$ as $f \in \mathscr{S}(\mathbb{R}^d)$. Let $C = (\rho_{0,0}(f) + \rho_{\tilde{n},0}(f))^p$. Thus we get,

$$\int_{\mathbb{R}^d} |f(x)|^p dx \leq C \int_{\mathbb{R}^d} \frac{1}{(1+|x|^{\tilde{n}})^p} dx = C\Big(\int_{|x|<1} \frac{1}{(1+|x|^{\tilde{n}})^p} dx + \int_{|x|>1} \frac{1}{(1+|x|^{\tilde{n}})^p} dx\Big)$$

$$\leq C\Big(\int_{|x|<1} \frac{1}{(1+|x|^{\tilde{n}})^p} dx + \int_{|x|>1} \frac{1}{(1+|x|^{\tilde{n}})} dx\Big) \leq C\Big(\int_{|x|<1} 1 dx + \int_{|x|>1} \frac{1}{(|x|^{\tilde{n}})} dx\Big)$$

which is finite as $p \in [1,\infty]$ and $1/(1+|x|) \leq 1$ for $|x| \geq 0$ □

3.2. **Tempered Distributions.** Tempered distributions are a special class of distributions. They are defined as linear functionals that are continuous on a larger space of test functions, which includes not only the space of smooth compactly supported functions, but also some spaces of rapidly decreasing functions.

Because tempered distributions are linear functionals that satisfy the continuity condition on a space of test functions, they are by definition distributions.

Definition 3.3. Tempered Distributions : The space $\mathscr{S}'\left(\mathbb{R}^d\right)$ of tempered distributions on $\mathbb{R}^d$ is the dual of $\mathscr{S}\left(\mathbb{R}^d\right)$, i.e. $f \in \mathscr{S}'\left(\mathbb{R}^d\right)$ means that f is a continuous linear functional on $\mathscr{S}(\mathbb{R}^d)$.

Theorem 3.4. A.6.1 in [3] *A linear functional f on $\mathscr{S}\left(\mathbb{R}^d\right)$ is a tempered distribution if and only if there exist $C>0$ and k,m integers such that*

$$|\langle f,\varphi\rangle| \leq C \sum_{|\alpha|\leq m, |\beta|\leq k} \rho_{\alpha,\beta}(\varphi), \quad \forall \varphi \in \mathscr{S}\left(\mathbb{R}^d\right)$$

or if there exist $C>0$ and $m\in\mathbb{N}_0$ such that

$$|\langle f,\varphi\rangle| \leq C \sum_{|\beta|_1\leq m} \left\|\left(1+|\cdot|^2\right)^m \partial^\beta \varphi\right\|_\infty \quad \forall \varphi \in \mathscr{S}\left(\mathbb{R}^d\right)$$

Note. In fact, every tempered distribution is a distribution, i.e $\mathscr{S}'\left(\mathbb{R}^d\right) \subset \mathscr{D}'\left(\mathbb{R}^d\right)$

Definition 3.4. Let $f_k, f \in \mathscr{S}'\left(\mathbb{R}^d\right)$ $\forall k \in \mathbb{N}$. We say that the sequence converges to the tempered distribution f if for every $\varphi \in \mathscr{S}$, the sequence $\langle f_k,\varphi\rangle$ converges to $\langle f,\varphi\rangle$.

Using the definition mentioned above we discuss some properties of tempered distribution like the sum of tempered distribution and what happens when we multiply by a constant. Consider $f,g \in \mathscr{S}'\left(\mathbb{R}^d\right), \alpha \in \mathbb{C}$ and consider φ_n to be a convergent sequence in $\mathscr{S}$ such that $\varphi_n \to \varphi$. We get $|\langle f+g,\varphi_n\rangle| \leq |\langle f,\varphi_n\rangle| + |\langle g,\varphi_n\rangle| \to |\langle f,\varphi\rangle| + |\langle g,\varphi\rangle|$ and $|\langle \alpha f,\varphi_n\rangle| \leq |\alpha| |\langle f,\varphi_n\rangle| \to |\alpha| |\langle f,\varphi\rangle|$.

Similar to Theorem (3.3) (Page 120 in [4]) we have the following result:

Theorem 3.5. *$L_p\left(\mathbb{R}^d\right) \subseteq \mathscr{S}'\left(\mathbb{R}^d\right)$ for $p\in[1,\infty]$.*

Proof. Again let $\beta = (0,0,\cdots 0) \in \mathbb{N}_0^d$. For every $\varphi \in \mathscr{S}\left(\mathbb{R}^d\right)$, we get $\left(\left|\left(1+x^{\tilde{n}}\right)\varphi(x)\right|\right)^p \leq \sup_{x\in\mathbb{R}^d}(|(1+x^{\tilde{n}})\varphi(x)|)^p = \left\|\left(1+x^{\tilde{n}}\right)\varphi\right\|_\infty^p = \left\|\varphi + x^{\tilde{n}}\varphi\right\|_\infty^p \leq (\rho_{0,0}(\varphi)+\rho_{\tilde{n},0}(\varphi))^p < \infty$.Therefore we have,

$$\left|\int_{\mathbb{R}^d} f(x)\varphi(x)dx\right| \leq \int_{\mathbb{R}^d} |f(x)||\varphi(x)| \frac{1+|x|^{\tilde{n}}}{1+|x|^{\tilde{n}}} dx \leq (\rho_{0,0}(\varphi)+\rho_{\tilde{n},0}(\varphi))^p \int_{\mathbb{R}^d} \frac{|f(x)|}{1+|x|^{\tilde{n}}} dx$$

Again let $C = (\rho_{0,0}(\varphi)+\rho_{\tilde{n},0}(\varphi))^p$. We saw in the proof of Theorem (3.3) that the function $1/\left(1+|x|^{\tilde{n}}\right)$ is in L_p. Therefore, let $\tilde{p}$ be such that $1/p+1/\tilde{p}=1$, by Hölder's inequality (B.1) we get

$$\left|\int_{\mathbb{R}^d} f(x)\varphi(x)dx\right| \leq C \int_{\mathbb{R}^d} \frac{|f(x)|}{1+|x|^{\tilde{n}}} dt \leq C\|f\|_p \left\|\frac{1}{1+|x|^{\tilde{n}}}\right\|_{\tilde{p}} < \infty$$

where $\tilde{p}$ is such that $1/p+1/\tilde{p}=1$ and it follows from Hölder's inequality (B.1). Therefore, $\int_{\mathbb{R}^d} f(x)\varphi(x)dx$ defines a distribution and let this be $f(\varphi)=\langle f,\varphi\rangle$. Linearity trivially follows and we show continuity. We have $\varphi_k(x) \to \varphi(x)$ and $|\varphi(x)| \leq C_{0,N}(1+|x|)^{-N}$ and we need to show $\langle f,\varphi_k\rangle \to \langle f,\varphi\rangle$. Well, by DCT (B.2),

$$\lim_{k\to\infty} \langle f,\varphi_k\rangle = \lim_{k\to\infty} \int_{\mathbb{R}^d} f(x)\varphi_k(x) = \int_{\mathbb{R}^d} \lim_{k\to\infty} f(x)\varphi(x) = \int_{\mathbb{R}^d} f(x)\varphi(x) = \langle f,\varphi\rangle$$

□

Remark. We could have also invoked Theorem (3.4) to prove the above but we wanted to show one example of proving a particular function is a tempered distribution using the definition. From here on we will just invoke Theorem (3.4).

Definition 3.5. The space $\mathscr{E}'\left(\mathbb{R}^d\right)$ of distributions with compact support on $\mathbb{R}^d$ is the linear space of all distributions in $\mathscr{D}'\left(\mathbb{R}^d\right)$ with compact support.

We say that $f_n \to f$ in $\mathscr{E}'\left(\mathbb{R}^d\right)$ if all $f, f_n \in \mathscr{E}'\left(\mathbb{R}^d\right)$ and $\langle f_n;\varphi\rangle \to \langle f;\varphi\rangle$ for all $\varphi \in C^\infty\left(\mathbb{R}^d\right)$. The dual spaces are nested as follows:

$$\mathscr{E}'\left(\mathbb{R}^d\right) \subseteq \mathscr{S}'\left(\mathbb{R}^d\right) \subseteq \mathscr{D}'\left(\mathbb{R}^d\right)$$

4. Fourier Transform

4.1. **Introduction.** The Fourier Transform is a powerful tool that helps us convert a function $f(x)$ defined in the (usually time) $x \in \mathbb{R}$ domain to another function $\widehat{f}(\xi)$ in the (frequency) ξ-domain which describes the frequency spectrum of the function f which is typically represented as a graph showing the amplitude or magnitude of each frequency component as a function of frequency. The Fourier Transform is a way to break down a signal, such as sound waves, into its individual frequency components, which are sine and cosine waves. This decomposition makes it easier to analyze and manipulate the signal and is used in many fields such as signal and image processing, noise reduction and feature extraction

Definition 4.1. Fourier Transform: Given $f \in \mathscr{S}\left(\mathbb{R}^d\right)$ the fourier transform $(\widehat{f}(\xi))$ is $\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi}dx$ and if $f \in \mathscr{S}'\left(\mathbb{R}^d\right)$, then $\langle \widehat{f}, \varphi\rangle := \langle f, \widehat{\varphi}\rangle$ for $\varphi \in \mathscr{S}(\mathbb{R}^d)$

Definition 4.2. Inverse Fourier Transform : Given $f \in \mathscr{S}\left(\mathbb{R}^d\right)$ the inverse fourier transform $(\check{f}(x))$ is $\check{f}(x) = \widehat{f}(-x) = \int_{\mathbb{R}^d} f(\xi)e^{2\pi i\xi\cdot x}d\xi$ and if $f \in \mathscr{S}'$ then $\langle \check{f}, \varphi\rangle = \langle f, \check{\varphi}\rangle$ for $\varphi \in \mathscr{S}(\mathbb{R}^d)$

We define the Fourier Series for now and will come back to it later,

Definition 4.3. Fourier Series: The Fourier series is a way of representing a periodic func- tion as a sum of sine and cosine functions with different amplitudes and frequencies. This enables us to express a large range of functions in terms of simple trigonometric functions. The fourier series and fourier transform are related in that the fourier transform can be thought of as the extension of the fourier series to non-periodic signals. Moreover, a periodic signal's fourier series coefficients can be calculated using the fourier transform, and vice versa. The Fourier series of a function $f \in L_1\left(\mathbb{T}^d\right)$ (where $\mathbb{T}^d$ (A.2) is the d-Torus) is defined as $\sum_{k\in\mathbb{Z}^d} \widehat{f}(k)e^{2\pi i k\cdot t}, t \in \mathbb{R}^d$ where the k-th Fourier coefficient $\widehat{f}(k)$ of f is $\widehat{f}(k) = \int_{\mathbb{T}^d} f(t)e^{-2\pi i k\cdot t}dt, k \in \mathbb{Z}^d$.

The following lemma tell us that the Fourier Transforms of a schwartz function and tempered distribution are schwartz function and tempered distribution respectively.

Lemma 1. A.6.3 in [3] *Let* $\varphi \in \mathscr{S}(\mathbb{R}^d)$, $f \in \mathscr{S}'(\mathbb{R}^d)$ *then* $\widehat{\varphi} \in \mathscr{S}(\mathbb{R}^d)$, $\widehat{f} \in \mathscr{S}'(\mathbb{R}^d)$.

The following important theorem is proved using the above lemma.

Theorem 4.1. A.6.3 in [3] *(i) The Fourier transform* $\mathscr{F} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R}), f \mapsto f$ *is a homeomorphism of* $\mathscr{S}(\mathbb{R})$ *onto itself and* $\mathscr{F}^{-1}$ *is its continuous inverse. ii) The Fourier transform* $\mathscr{F} : \mathscr{S}'(\mathbb{R}) \to \mathscr{S}'(\mathbb{R}), f \mapsto \widehat{f}$ *is a homeomorphism of* $\mathscr{S}'(\mathbb{R})$ *onto itself and* $\mathscr{F}^{-1}$ *is its continuous inverse. Moreover, if* $f_n \to f$ *in* $\mathscr{S}'\left(\mathbb{R}^d\right)$ *as* $n \to \infty$*, then* $\widehat{f}_n \to \widehat{f}$ *in* $\mathscr{S}'\left(\mathbb{R}^d\right)$ *as* $n \to \infty$.

In other words, because f is smooth, its Fourier transform is rapidly decreasing, and because f is rapidly decreasing, its Fourier transform is smooth. Now since our project deals with translates of functions, the following is an obvious consequence of the above definitions,

Proposition 4.2. *Let* $\varphi \in \mathscr{S}\left(\mathbb{R}^d\right)$. *Then, if* $k \in \mathbb{R}^d$, *then* $\widehat{\varphi}(x-k) = e^{-2\pi i\xi\cdot k}\widehat{\varphi}(\xi)$ *and if* $\varphi, \phi \in \mathscr{S}\left(\mathbb{R}^d\right)$, *then* $\int_{\mathbb{R}^d} \widehat{\varphi}(x))\phi(x)dx = \int_{\mathbb{R}^d} \varphi(x)\widehat{\phi}(x))dx$.

Proof. Let $\varphi \in \mathscr{S}\left(\mathbb{R}^d\right)$ and $k \in \mathbb{R}^d$ and define $\tilde{\varphi} := \varphi(x-k)$. Then we have

$$\widehat{\tilde{\varphi}}(x) = \int_{\mathbb{R}^d} \tilde{\varphi}(x)e^{-2\pi i x.\xi}dx = \int_{\mathbb{R}^d} \varphi(x-k)e^{-2\pi i x.\xi}dx = e^{-2\pi i\xi.k}\int_{\mathbb{R}^d} \varphi(x-k)e^{-2\pi i(x-k).\xi}dx = e^{-2\pi i\xi.k}\widehat{\varphi}(\xi)$$

with the change of variable $x \leftarrow x-k$.

The second part of the proposition is a direct consequence of Fubini's Theorem (B.3) and we can use Fubini's by Theorem (3.3). ☐

Note. The second part of the above proposition also helps us justify the definition of the fourier transform of $f \in \mathscr{S}(\mathbb{R}^d)$.

Note. We note that the above proposition also helps us justify the definition of the fourier transform of distributions $f \in \mathscr{S}'$ and also translates of $f \in \mathscr{S}'$. We claim for every $f \in \mathscr{S}'(\mathbb{R}^d)$, and $k \in \mathbb{R}^d$, $\langle \widehat{f}(x-k), \varphi(x)\rangle = \langle f(x), \widehat{\varphi}(x+k)\rangle$. Let $\tilde{f} := f(x-k)$,we have,
$\langle \widehat{\tilde{f}}, \varphi\rangle = \int_{\mathbb{R}^d} \widehat{\tilde{f}}\varphi dx = \int_{\mathbb{R}^d} (\int_{\mathbb{R}^d} f(x-k)e^{-2\pi i\xi x}dx)\varphi(x)dx = \int_{\mathbb{R}^d} (\int_{\mathbb{R}^d} f(x-k)e^{-2\pi i\xi x}dx)\varphi(x)dx = \int_{\mathbb{R}^d} (\int_{\mathbb{R}^d} f(x)\varphi(x+k)e^{-2\pi i\xi(x+k)}dx)dx = (\int_{\mathbb{R}^d} f(x)\widehat{\varphi}(x+k)dx = \langle f(x), \widehat{\varphi}(x+k)\rangle$ □

Example 4.1. For $x_0 \in \mathbb{R}^d$, we define $\langle \delta_k, \varphi\rangle := \varphi(k)$ and we trivially have $\delta_k \in \mathscr{S}'(\mathbb{R}^d)$. Also,

$$\left\langle \widehat{\delta_k}, \varphi \right\rangle = \langle \delta_k, \widehat{\varphi}\rangle = \widehat{\varphi}(k) = \int_{\mathbb{R}^d} \varphi(\xi)e^{-2\pi i\xi\cdot k}d\xi = \left\langle e^{-2\pi i\xi\cdot k}, \varphi\right\rangle, \quad \delta_k = \delta(x_0 - k)$$

Proposition 4.3. (Fourier inversion) 3.2.5 in [4] *Suppose that $f \in L_1(\mathbb{T}^d)$ and that $\sum_{m\in\mathbb{Z}^d} |\widehat{f}(m)| < \infty$. Then $f(x) = \sum_{m\in\mathbb{Z}^d} \widehat{f}(m)e^{2\pi i m\cdot x}$ a.e., and therefore f is almost everywhere equal to a continuous function.*

Remark. A.5.4 in [3] If f, $\widehat{f}$ are in $L_1(\mathbb{R}^d)$, then $f(x) = \int_{\mathbb{R}^d} \widehat{f}(\xi)e^{2\pi i\xi\cdot x}d\xi$, a.e $x \in \mathbb{R}^d$. Therefore, both $f, \widehat{f} \in L_1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$ and if $\varphi \in \mathscr{S}(\mathbb{R}^d)$, $f \in \mathscr{S}'(\mathbb{R}^d)$ then $\check{\widehat{\varphi}} = \widehat{\check{\varphi}} = \varphi$ and $\check{\widehat{f}} = \widehat{\check{f}} = f$.

The Fourier Inversion formula shows that the original function can be recovered from its Fourier transform by the inverse Fourier transform formula.

4.2. **Poisson Summation formula.** The motivation for the Poisson Summation formula (PSF) comes from the Fourier series, which expresses a periodic function as a sum of sinusoidal functions. The Fourier transform is a generalization of the Fourier series to non-periodic functions. The PSF relates the Fourier transform of a function on the real line to its values on the integers. It states that if a function and its Fourier transform satisfy certain conditions, then the sum of the values of the function on the integers is equal to the sum of the values of its Fourier transforms on the integers. This result is remarkable because it relates a continuous function on the real line to a discrete sequence of values on the integers. The PSF provides a way to calculate the values of a function at integer points using its Fourier transform, which may be easier to compute or manipulate.

Theorem 4.4. (PSF) A.5.7 in [3], 3.2.8 in [4] *Suppose that $f, \widehat{f} \in L_1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ and $f \in \mathscr{S}(\mathbb{R}^d)$ i.e $|f(x)| \le C(1+|x|)^{-n-\varepsilon}$, $\forall x \in \mathbb{R}^d$, for some $C, \varepsilon > 0$ and whose Fourier transform $\widehat{f}$ restricted on $\mathbb{Z}^d$ satisfies $\sum_{m\in\mathbb{Z}^d} |\widehat{f}(m)| < \infty$. Then we have the relation,*

$$\sum_{k\in\mathbb{Z}^d} \widehat{f}(k)e^{2\pi i k\cdot x} = \sum_{k\in\mathbb{Z}^d} f(x+k),\ \forall x \in \mathbb{R}^d$$

We follow the proof 3.2.8 in [4] but add and justify any necessary details.

Proof. Define a function $f^{\text{per}}(\cdot)$ on $\mathbb{T}^d$, $f^{\text{per}}(x) = \sum_{k\in\mathbb{Z}^d} f(x+k)$. This is clearly 1- periodic as $f^{\text{per}}(x+1) = \sum_{k\in\mathbb{Z}^d} f(x+k+1) = \sum_{\tilde{k}\in\mathbb{Z}^d} f(x+\tilde{k}) = f^{\text{per}}(x)$ where $\tilde{k} = k+1$. By the assumption that $|f(x)| \le C(1+|x|)^{-n-\varepsilon}$, $\forall x \in \mathbb{R}^d$, the above series converges absolutely and uniformly. Hence, $f^{\text{per}} \in C_0(\mathbb{T}^d) \subset L_1(\mathbb{T}^d)$. In fact we justify that $\|f^{\text{per}}\|_{L^1([0,1]^d)} = \|f\|_{L^1(\mathbb{R}^d)}$.

$$\|f^{\text{per}}\|_{L_1([0,1]^d)} = \int_{\mathbb{T}^d} \left|\sum_{k\in\mathbb{Z}^d} f(x+k)\right| dx \le \int_{\mathbb{T}^d} \sum_{k\in\mathbb{Z}^d} |f(x+k)|\, dx = \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{T}^d} |f(x+k)|\, dx$$

which is equal to $\sum_{k\in\mathbb{Z}^d} \int_{[-\frac{1}{2},\frac{1}{2}]^d - k} |f(x)| = \int_{\mathbb{R}^d} |f(x)|dx = \|f\|_{L^1(\mathbb{R}^d)}$. The other direction is similar by using the roles of f^{per} and f reversed. Now we prove that the sequence of the Fourier coefficients of f^{per} equals to the restriction of the Fourier transform of f on $\mathbb{Z}^d$. This follows from :

$$\widehat{f^{\text{per}}}(m) = \int_{\mathbb{T}^d} \sum_{k\in\mathbb{Z}^d} f(x+k)e^{-2\pi i m\cdot x} = \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{T}^d} f(x+k)e^{-2\pi i m\cdot x} = \sum_{k\in\mathbb{Z}^d} \int_{[-\frac{1}{2},\frac{1}{2}]^d - k} f(x)e^{-2\pi i m\cdot x}$$

which is equal to $\int_{\mathbb{R}^d} f(x)e^{-2\pi i m\cdot x}dx = \widehat{f}(m)$ where $x \leftarrow x+k$. Now we justify the interchange of sum and integral in the above proofs. For any $x \in \mathbb{T}^d = [0,1]^d$, we note that $0 \le |x| \le \sqrt{n}$ as $|x| := (|x_1|^2 + \cdots + |x_n|^2)^{1/2}$ for $x \in \mathbb{T}^d$. Therefore, by $|f(x)| \le C(1+|x|)^{-n-\varepsilon}$, $\forall x \in \mathbb{R}^d$, and by the Weierstrass M-test of uniform convergence of series,

$$f^{\text{per}}(x) = \sum_{k\in\mathbb{Z}^d} f(x+k) \le \sum_{k\in\mathbb{Z}^d} \frac{1}{(1+|k+x|)^{n+\varepsilon}} \le \sum_{k\in\mathbb{Z}^d} \frac{(1+\sqrt{n})^{n+\varepsilon}}{(1+\sqrt{n}+|k+x|)^{n+\varepsilon}} \le \sum_{k\in\mathbb{Z}^d} \frac{C_{n,\varepsilon}}{(1+|k|)^{n+\varepsilon}},$$

which is $< \infty$ and we used $|k+x| \ge |k| - |x| \ge |k| - \sqrt{n}$. This calculation also shows that f^{per} is the sum of a uniformly convergent series of continuous functions on $[0,1]^d$, thus it is itself continuous. Hence, Prop (4.3) applies, and given the fact that f^{per} is continuous, it yields,

$$\sum_{m\in\mathbb{Z}^d} \widehat{f}(m)e^{2\pi i m\cdot x} = \sum_{k\in\mathbb{Z}^d} f(x+k)$$

for all $x \in \mathbb{T}^d$ and, by periodicity, this holds for all $x \in \mathbb{R}^d$. □

Remark. In fact, if we put $x = 0 \in \mathbb{R}^d$ in the above equation we get

$$\sum_{m\in\mathbb{Z}^d} \widehat{f}(m) = \sum_{k\in\mathbb{Z}^d} f(k)$$

which is a more clear representation of how the PSF relates a function defined on the real line to the Fourier series and this gives us a deep insight into the structure of functions and their Fourier series, allowing us to understand their properties and behavior in a more fundamental way.

Theorem 4.5. *(the Poisson Summation Formula for Distributions)* A.6.5 in [3] *Let f be a compactly supported distribution on $\mathbb{R}^d$. For $\zeta \in \mathbb{C}^d$,*

$$\sum_{k\in\mathbb{Z}^d} f(x-k)e^{-i\zeta(x-k)} = \sum_{k\in\mathbb{Z}^d} \widehat{f}(\zeta+2\pi k)e^{i2\pi k\cdot x}$$

in the sense of distributions.

4.3. Parseval's Relation and Plancherel's Identity.

Theorem 4.6. 2.2.4 in [4] *Given f, g in $\mathscr{S}(\mathbb{R}^d)$, we have*

(1) (Parseval's relation) $\int_{\mathbf{R}^d} f(x)\overline{g(x)}dx = \int_{\mathbb{R}^d} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi,$

(2) (Plancherel's identity) $\|f\|_{L_2} = \|\widehat{f}\|_{L_2} = \left\|\check{f}\right\|_{L_2}$

Proof. Define $\tilde{f}(x) = f(-x)$. Now we have by the definition (4.1) we have

$$\widehat{\overline{f}}(\xi) = \int_{\mathbb{R}^d} \overline{f}(x)e^{-2\pi i x\cdot\xi}dx = \int_{\mathbb{R}^d} \overline{f(x)e^{2\pi i x\cdot\xi}}dx = \overline{\int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi}dx} = \overline{\widehat{f}(-\xi)} = \overline{\widetilde{\widehat{f}}}(\xi)$$

Using this and by Proposition (4.2) we have $\int_{\mathbb{R}^d} f(x)\widehat{h}(x)dx = \int_{\mathbb{R}^d} \widehat{f}(x)h(x)dx$, definition (4.2), remark (4.3) and by using the fact that $\widehat{\widehat{f}} = \widetilde{\widetilde{f}}$, we let $h = \overline{\widehat{g}}$ and we get the following,

$$\widehat{h} = \widehat{\overline{\widehat{g}}} = \overline{\widetilde{\widehat{\widehat{g}}}} = \overline{\widetilde{\widetilde{\widetilde{g}}}} = \overline{\widetilde{\widetilde{g}}} = \overline{\widetilde{g}} = \overline{g}$$

which gives us (1). Now (2) is a straightforward consequence of (1). □

Similar to the above theorem which is for functions defined in the Schwartz Space we have the following proposition which is defined for functions in the $d-$Torus.

Proposition 4.7. 3.2.7 in [4] *The following are valid for $f, g \in L_2(\mathbb{T}^d)$:*

(1) (Parseval's relation) $\int_{\mathbb{T}^d} f(t)\overline{g(t)}dt = \sum_{m\in\mathbb{Z}^d} \widehat{f}(m)\overline{\widehat{g}(m)}.$

(2) (Plancherel's identity) $\|f\|_{L_2}^2 = \sum_{m\in\mathbb{Z}^d} |\widehat{f}(m)|^2$

Parseval's relation and Plancherel's identity are two significant results in mathematics that relate to the properties of functions and their Fourier transforms. Both results have critical applications in many areas of mathematics and science, including signal processing, quantum mechanics, and harmonic analysis and they will prove useful to us when we discuss stability of functions.

5. Linear Independence

For $v = \{v_k\}_{k\in\mathbb{Z}^d} \in \ell(\mathbb{Z}^d)$ and a compactly supported distribution f, we say $\{f(\cdot - k)\}_{k\in\mathbb{Z}^d}$ is called $\ell(\mathbb{Z}^d)$ - linearly independent or (*globally*)-linearly independent (5.2.2 in [3]) if $\sum_{k\in\mathbb{Z}^d} v_k f(\cdot - k) = 0 \implies v_k = 0 \;\forall\; k \in \mathbb{Z}^d$. To show this for any $f \in \mathscr{E}'(\mathbb{R}^d)$ we need to prove the claim below.

Linear Independence is important for integer shifts of compact support functions because it allows for stability and shift invariance i.e, when the functions are linearly independent, the shifted versions are also linearly independent, which means that the shifted functions retain their unique characteristics and can be distinguished from one another. This property is handy in signal processing and image processing.

Claim 1. 2.3.8 in [4] *Let $(c_k)_{k\in\mathbb{Z}^d} \in \ell(\mathbb{Z}^d)$ such that $|c_k| \le A(1+|k|)^M$ for all k and some fixed M and $A > 0$. Let δ_k denote Dirac mass at the integer k. Then the sequence $\sum_{|k|\le n} c_k\delta_k$ converges to some tempered distribution f in $\mathscr{S}'\left(\mathbb{R}^d\right)$ as $n \to \infty$. Also $\widehat{f}$ is the $\mathscr{S}'$ limit of the sequence of functions $f_n(\xi) = \sum_{|k|\le n} c_k e^{-2\pi i\xi\cdot k}$.*

Proof. Let $\varphi(x) \in \mathscr{S}(\mathbb{R}^d)$, $f_n(x) := \sum_{|k|\le n} c_k\delta_k$ and show $\lim_{n\to\infty}\langle f_n, \varphi\rangle = \langle f, \varphi\rangle = f \in \mathscr{S}'(\mathbb{R}^d)$.

$$|\langle f_n, \varphi\rangle| = \left|\left\langle \sum_{|k|\le n} c_k\delta_k, \varphi(x)\right\rangle\right| \le \sum_{|k|\le n} |c_k|\,|\langle\delta_k, \varphi(x)\rangle| \le \sum_{|k|\le n} A(1+|k|)^M|\varphi(k)|$$

$$\le \sum_{|k|\le n} A(1+|k|)^M \left|\varphi(k)\frac{(1+|k|)^{d+1}}{(1+|k|)^{d+1}}\right| \le A\sup_{k\in\mathbb{Z}^d}|\varphi(k)(1+|k|)^{M+d+1}| \sum_{k\in\mathbb{Z}^d}\left|(1+|k|)^{-d-1}\right|$$

Now by Weierstrass M -test of uniform convergence of series, $\sum_{k\in\mathbb{Z}^d}\left|(1+|k|)^{-d-1}\right|$ converges and let this converge to some $\tilde{A}$. Let $A' = A\tilde{A}$. Therefore we get, $A\sup_{k\in\mathbb{Z}^d}|\varphi(k)(1+|k|)^{M+d+1}| = A'\left\|(1+|k|)^{M+d+1}\varphi\right\|_\infty < \infty$ and we get,

$$A'\left\|(1+|k|)^{M+d+1}\varphi\right\|_\infty = A'\left\|\sum_{j=0}^{M+d+1}\binom{M+d+1}{j}|k|^j\varphi\right\|_\infty \le A'\sum_{j=0}^{M+d+1}\binom{M+d+1}{j}\rho_{j,0}$$

$\le A'(M+d+1)\binom{M+d+1}{\lfloor\frac{M+d+1}{2}\rfloor}\sum_{j=0}^{M+d+1}\rho_{j,0} < \infty$. From this and by the discrete version of DCT (B.2) we get $\lim_{n\to\infty}\langle f_n, \varphi\rangle = \langle f, \varphi\rangle = f$. Let $\beta = (0, 0, \cdots 0) \in \mathbb{N}_0^d$. By Theorem (3.4), we get that $\langle f, \varphi\rangle \in \mathscr{S}'(\mathbb{R}^d)$. Now recall $\langle\widehat{\delta_k}, \varphi\rangle = \left\langle e^{-2\pi i\xi\cdot k}, \varphi\right\rangle$ by example (4.1). Since $|c_k| \le A(1+|k|)^M$ we get $|f_n(\xi)| = \left|\sum_{|k|\le n} c_k e^{-2\pi i\xi\cdot k}\right| \le \sum_{|k|\le n}|c_k e^{-2\pi i\xi\cdot k}| = \sum_{|k|\le n}|c_k| \le \sum_{|k|\le n} A(1+|k|)^M = \mathcal{O}(n^M)$ which shows polynomial growth of $f_n(\xi)$ and shows us that its bounded by Cn^M for some $C > 0$ by (A.6). Therefore we can take Fourier transform on both sides of $f_n(x) = \sum_{|k|\le n} c_k\delta_k$ to get $f_n(\xi) = \sum_{|k|\le n} c_k e^{-2\pi i\xi\cdot k}$. Thus, by Lemma (L.1) and Theorem (4.1) we get the desired result. ☐

Remark. The reason we need the above claim and the compactly supported condition is that it helps us define the infinite sum in the compact distribution space which is endowed with the weak topology defined in Definition (3.1) , it helps us justify taking the Fourier transform of the infinite sum f and also if $f \in \mathscr{E}'(\mathbb{R}^d)$ then in fact by the Schwartz's Paley-Weiner Theorem A.6.4 in [3], $\widehat{f}$ defined on $\mathbb{R}^d$ is actually an entire function on $\mathbb{C}^d$!!

We finally come to the main results of the thesis which involves proving Linear Independent of the Integer Shift of Compactly Supported Distributions.

Lemma 2. *Let $\widehat{f} = \sum_{k\in\mathbb{Z}^d} c_k e^{-2\pi i\xi\cdot k} \in \mathscr{E}'(\mathbb{R}^d)$. If $\widehat{f} = 0$ then $c_k = 0 \;\;\forall k \in \mathbb{Z}^d$ where $(c_k)_{k\in\mathbb{Z}^d} \in \ell(\mathbb{Z}^d)$ such that $|c_k| \le A(1+|k|)^M$ for all k and some fixed M and $A > 0$.*

Proof. To prove linear independence of $\widehat{f}$ we use the Function Independence Theorem (Prop 3.1 in [5]). Consider the finite sum $f_n(\xi) := \sum_{1\leq k\leq n} c_k e^{-2\pi i\xi\cdot k}$. We first show this is linearly independent. It follows that $\sum_{|k|\leq n} c_k e^{-2\pi i\xi\cdot k}$ is linearly independent. For simplicity let $m = -2\pi i\xi$ and consider the Wronskian of $\{e^m, e^{2m}, \cdots e^{nm}\}$. Denote this by $\mathscr{W}(\{e^m, e^{2m}, \cdots e^{nm}\})$. We have, $\mathscr{W}(\{e^m, e^{2m}, \ldots e^{nk}\}) =$

$$\begin{vmatrix} e^m & e^{2m}\cdots e^{nm} \\ e^m & 2e^{2m}\cdots ne^{nm} \\ e^m & 2^2e^{2m}\cdots n^2e^{nm} \\ \vdots & \ddots \\ \vdots & \ddots \\ e^m & 2^ne^{2m}\cdots n^{n-1}e^{nm} \end{vmatrix} = \begin{vmatrix} 1 & 1 & \cdots & \cdots 1 \\ 1 & 2 & \cdots & \cdots n \\ 1 & 2^2 & \cdots & \cdots n^2 \\ \vdots & \ddots & \cdots & \\ \vdots & \ddots & \cdots & \\ 1 & 2^n & \cdots & \cdots n^{n-1} \end{vmatrix} \begin{vmatrix} e^m & \cdots & \cdots \\ 0 & e^{2m}\ldots 0 & \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ 0 & 0 & \cdots e^m \end{vmatrix}$$

where we used $|AB| = |A||B|$. We can do this as both $|A|, |B| \neq 0$, to see this let A be the left matrix and B be the diagonal matrix. Clearly $|B| = \prod_{1\leq i\leq n} e^{mi} \neq 0$ as $e^{mi} \neq 1$ $\forall i \in \mathbb{Z}^d$. Now consider A,

$$\begin{vmatrix} 1 & 1 & \cdots & \cdots 1 \\ 1 & 2 & \cdots & \cdots n \\ 1 & 2^2 & \cdots & \cdots n^2 \\ \vdots & \ddots & \cdots & \\ \vdots & \ddots & \cdots & \\ 1 & 2^n & \cdots & \cdots n^{n-1} \end{vmatrix} = \prod_{1\leq i<j\leq n}(a_j - a_i) = (n-1)!(n-2)!\ldots 1 \neq 0$$

where $a_i, a_j \in \{1, 2, \ldots n\} \ni a_i \neq a_j$. We get the above computation as A is essentially just the Vandermonde matrix [6] whose determinant is nonzero if and only if all a_i, a_j are not equal. Therefore $\mathscr{W}(\{e^m, e^{2m}, \ldots e^{nk}\}) \neq 0$ and we get that $\{e^m, e^{2m}, \ldots e^{nk}\}$ is a linearly independent set which implies that $\sum_{1\leq k\leq n} c_k e^{-2\pi i\xi\cdot k} = 0 \implies c_k = 0$ $\forall k \in \mathbb{Z}^d$. Following a similar calculation above we also get that $\sum_{|k|\leq n} c_k e^{-2\pi i\xi\cdot k} = 0 \implies c_k = 0$ $\forall k \in \mathbb{Z}^d$. Now from the above claim (C.1) we know that the infinite sum $\sum_{k\in\mathbb{Z}^d} c_k e^{-2\pi i\xi\cdot k}$ is defined and using the fact that $\mathscr{E}' \subseteq \mathscr{S}'$. Since we showed that for arbitrary finite n, $f_n(\xi)$ is linearly independent, we can conclude that the infinite sum $\widehat{f}$ is linearly independent. □

The following Theorem proves Linear Independence for integer shifts of functions and is a direct consequence of the above lemma and by the definition of the Fourier Transform. A similar result is shown as Proposition 9.6.2 in [7] which shows linear independence for a function $f \in L_2(\mathbb{R})\backslash\{0\}$.

Theorem 5.1. *Let $0 \not\equiv f \in \mathscr{E}'(\mathbb{R}^d)$. Then for $x \in \mathbb{R}^d$, $\sum_{k\in\mathbb{Z}^d} c_k f(x-k) = 0$ implies $c_k = 0$ $\forall k \in \mathbb{Z}^d$ where $(c_k)_{k\in\mathbb{Z}^d} \in \ell(\mathbb{Z}^d)$ such that $|c_k| \leq A(1+|k|)^M$ for all k and some fixed M and $A > 0$.*

Proof. For some $x \in \mathbb{R}^d$ consider $\sum_{k\in\mathbb{Z}^d} c_k f(x-k) = 0$. Taking Fourier Transform on both sides (justified above claim (C.1)), we get,

$$\sum_{k\in\mathbb{Z}^d} c_k f(x-k) = 0 \implies \sum_{k\in\mathbb{Z}^d} c_k e^{-2\pi i\xi\cdot k}\widehat{f}(\xi) = 0 \implies \widehat{f}(\xi)\sum_{k\in\mathbb{Z}^d} c_k e^{-2\pi i\xi\cdot k} = 0$$

by proposition (4.2). Now $\widehat{f}(\xi) \not\equiv 0$ as $f \not\equiv 0$ as theorem (4.1) tells us that $\widehat{f}$ is an homeomorphism. Therefore we must have $\sum_{k\in\mathbb{Z}^d} c_k e^{-2\pi i\xi\cdot k} = 0$. By Lemma L.2, we get that $\sum_{k\in\mathbb{Z}^d} c_k f(x-k) = 0$ implies $c_k = 0$. □

Definition 5.1. Semi-Discrete Convolution: Semi-Discrete convolution is a mathematical operation that is used to combine two sequences of values (signals) in a specific way. It basically allows one to shift and multiply one sequence by the other, and then sum the resulting values. For $c = \{v_k\}_{k\in\mathbb{Z}} \in \ell(\mathbb{Z}^d)$ and a compactly supported distribution f, we define the semi-discrete convolution to be $(v * f)(\cdot) := \sum_{k\in\mathbb{Z}^d} v(k)f(\cdot - k)$.

Theorem 5.2. *Let $f \in \mathscr{E}'\left(\mathbb{R}^d\right)$ and consider the sets $V = \{v \in \ell(\mathbb{Z}) : v * f = 0\}$ and $S = \{\xi \in \mathbb{C}^d : \widehat{f}(\xi + 2\pi k) = 0, \forall k \in \mathbb{Z}^d\}$. We then have $\{e^{i\xi\cdot k}\}_{k\in\mathbb{Z}^d} \in V$ if and only if $\xi \in S$ for some $\xi \in \mathbb{C}^d$.*

Remark. The set V can be considered as the Kernel of the space of shift of compactly supported distributions and the theorem above relates this Kernel to the Linear Independence of the Fourier transform of the distribution and this is Theorem 5.2.1 in [3] and Theorem 1.1 in [8]

Proof. ($\Longleftarrow$) Consider the set S as defined above. If $\exists\, \xi \in \mathbb{C}^d \ni \xi \in S$ then $\widehat{f}(\xi + 2\pi k) = 0,\ \forall k \in \mathbb{Z}^d$. Now by PSF for distributions (4.5), we get

$$\sum_{k\in\mathbb{Z}} \widehat{f}(\xi + 2\pi k)e^{i2\pi x\cdot k} = \sum_{k\in\mathbb{Z}} f(x-k)e^{-i\xi(x-k)} = \sum_{k\in\mathbb{Z}} f(x-k)e^{-i\xi\cdot x}e^{i\xi\cdot k}.$$

We have $\widehat{f}(\xi + 2\pi k) = 0,\ \forall k \in \mathbb{Z}^d$. Thus,

$$\sum_{k\in\mathbb{Z}} f(x-k)e^{-i\xi\cdot x}e^{i\xi\cdot k} = 0 \implies e^{-i\xi\cdot x}\sum_{k\in\mathbb{Z}} f(x-k)e^{i\xi\cdot k} = 0$$

Since, $e^{-i\xi\cdot x} \neq 0 \implies \sum_{k\in\mathbb{Z}} f(x-k)e^{i\xi\cdot k} = 0$. Which is essentially $e^{i\xi k} * f = 0$. Take $v(k) = e^{i\xi\cdot k}$. ($\Longrightarrow$) Let $v(k) = e^{i\xi\cdot k} \in V$. Therefore we get $e^{i\xi\cdot k} * f = 0$. Multiply both sides by $e^{i\xi\cdot x}$. Using PSF for distributions (4.5) and going in the reverse direction similar to the above direction by taking $c_k = \widehat{f}(\xi + 2\pi k)$ and using the previous Theorem (5.1) we get the claim. □

6. Stability

The idea of function stability refers to how a function behaves when its input or parameters are perturbed somewhat. A function is considered to be stable if it remains confined or does not vary considerably when its input or parameters are changed slightly. In other words, if a function is stable, slight changes in its input or parameters will not result in large changes in its output. This is a significant notion in many fields of mathematics, including numerical analysis, differential equations, and control theory, where mathematical models must be stable.

For $1 \le p \le \infty$, we say that the integer shift of a compactly supported distribution $f \in L_p(\mathbb{R}^d)$ is stable in $L_p(\mathbb{R}^d)$ if there exist positive constants C_1 and C_2 such that for all $\{v_k\}_{k\in\mathbb{Z}^d} \in \ell_p$ we have,

$$C_1\|\{v_k\}\|_{\ell_p(\mathbb{Z}^d)} \le \left\|\sum_{k\in\mathbb{Z}^d} v_k f(\cdot - k)\right\|_{L_p(\mathbb{R}^d)} \le C_2\|\{v_k\}\|_{\ell_p(\mathbb{Z}^d)}$$

The proof for showing the stability of integer shifts of functions in $L_p(\mathbb{R})$ (Theorem 5.3.4 in [3]) is done via characterization which states that if f is a compactly supported function distribution on $\mathbb{R}^d$ and let $\widehat{f}$ be it's Fourier transform then, the integer shift of f is stable if and only if $\widehat{f}$ does not possess in $\mathbb{R}^d$ any $2\pi k$-periodic zeros, i.e., the set $S' = \left\{\xi \in \mathbb{R}^d : \hat{f}(\xi + 2\pi k) = 0,\ \forall k \in \mathbb{Z}\right\}$ is empty.

Now comparing the set S in Theorem(5.2) with the above set S' we get that linear independence implies stability. Since the study in general Banach Spaces is much more involved we restrict our study and discussion to the Hilbert Space and briefly talk about a certain type of sequences called the Riesz Sequences which are actually stable and mention some notable results for the same.

Let $\mathcal{H}$ be a Hilbert space. The space $L_2\left(\mathbb{R}^d\right)$ is a Hilbert space with the inner product given by

$$\langle f, g\rangle := \int_{\mathbb{R}^d} f(x)\overline{g(x)}\mathrm{d}x, \quad f, g \in L_2\left(\mathbb{R}^d\right),$$

where $\overline{g(x)}$ denotes the complex conjugate of $g(x)$. The inner product of two elements f and g in $\mathcal{H}$ is denoted by $\langle f, g\rangle$. If $\langle f, g\rangle = 0$, then we say that f is orthogonal to g. The norm of an element f in $\mathcal{H}$ is given by $\|f\| := \sqrt{\langle f, f\rangle}$.

Definition 6.1. Riesz Sequence A sequence $(f_k)_{k\in\mathbb{Z}^d}$ in $\mathcal{H}$ is called a Riesz sequence if there exist two positive constants A and B such that the inequalities

$$A\left\|\{c_k\}\right\|_{\ell_2} \leq \left\|\sum_{k\in\mathbb{Z}^d} c_k f_k\right\|_{L_2(\mathbb{R}^d)} \leq B\left\|\{c_k\}\right\|_{\ell_2}$$

hold true for every finite sequence $(c_k)_{k\in\mathbb{Z}^d}$. If $(f_k)_{k\in\mathbb{Z}^d}$ is a Riesz sequence in $\mathcal{H}$, and if the linear span of $\{f_k : k\in\mathbb{Z}^d\}$ is dense in $\mathcal{H}$, then $(f_k)_{k\in\mathbb{Z}^d}$ is a Riesz basis of $\mathcal{H}$ and where,

$$\left\|\{c_k\}\right\|_{\ell_2}^2 = \sum_{k\in\mathbb{Z}^d} |c_k|^2 < \infty$$

We now recall the Fourier Series we introduced in (A.2). For a function $f\in L_1(\mathbb{T}^d)$ (where $\mathbb{T}^d$ (A.2) is the d-Torus) the Fourier Series is defined as $\sum_{k\in\mathbb{Z}^d}\widehat{f}(k)e^{-2\pi ik\cdot t}, t\in\mathbb{R}^d$, where the k-th Fourier coefficient $\widehat{f}(k)$ of f is $\widehat{f}(k)=\int_{\mathbb{T}^d} f(t)e^{2\pi ik\cdot t}dt,\ k\in\mathbb{Z}^d$ and let $c_k=\widehat{f}(k)$ for simplicity.

Definition 6.2. The bracket product of two functions $f,g\in L_2(\mathbb{R}^d)$ is defined as follows:

$$[f,g](\xi) := \sum_{k\in\mathbb{Z}^d} \hat{f}(\xi+2\pi k)\overline{\hat{g}(\xi+2\pi k)}, \quad \xi\in\mathbb{R}^d$$

and note that $[f,f](\xi) := \sum_{k\in\mathbb{Z}^d}|\hat{f}(\xi+2\pi k)|^2$.

Now we actually have that $[f,g]$ is a periodic function on $\mathbb{R}^d$ and is integrable over $\mathbb{T}^d$ and its Fourier coefficients [11] are given by

$$\begin{aligned} c_k &:= \int_{\mathbb{T}^d}[f,g](\xi)e^{2\pi ik\cdot\xi}d\xi = \frac{1}{(2\pi)^s}\int_{T^d}\sum_{\tilde{k}\in\mathbb{Z}^s}\hat{f}(\xi+2\tilde{k}\pi)\overline{\hat{g}(\xi+2\tilde{k}\pi)}e^{2\pi ik\cdot\xi}d\xi \\ &= \int_{\mathbb{R}^d}\hat{f}(\xi)\overline{\hat{g}(\xi)e^{-2\pi ik\cdot\xi}}d\xi = \langle f, g(\cdot-k)\rangle,\ k\in\mathbb{Z}^d, \end{aligned}$$

where we used Parseval's relation (4.6). Now Plancherel's Identity (4.6) gives us,

$$\sum_{k\in\mathbb{Z}^d}|\langle f,g(\cdot-k)\rangle|^2 = \int_{\mathbb{T}^d}|[f,g](\xi)|^2d\xi.$$

Thus, f is orthogonal to $g(\cdot-k)$ for all $k\in\mathbb{Z}^d$ if and only if $[f,g](\xi)=0$ for almost every $\xi\in\mathbb{R}^d$. Let A and B be the essential infimum and essential supremum of $[f,f]$ over $\mathbb{T}^d$. Then $(f(\cdot-k))_{k\in\mathbb{Z}^d}$ is a Riesz sequence in $L_2(\mathbb{R}^d)$ if and only if $B<\infty$ (respectively, $0<A\leq B<\infty$). In particular, we have,

Theorem 6.1. 3.24 in [10] *For any function $f\in L_2(\mathbb{R}^d)$ and constants $0<A\leq B<\infty$, the following two statements are equivalent:*
(i) $\{f(\cdot-k)\}_{k\in\mathbb{Z}^d}$ is a Riesz sequence with Riesz bounds A and B
(ii) The Fourier transform $\hat{f}$ of f satisfies

$$A\leq[f,f](\xi)\leq B,\ \textit{a.e. for some } \xi\in\mathbb{R}^d$$

The following is a direct consequence of the above definitions and theorems which is essentially the characterization of linear independence property we mentioned at the beginning of our introduction to stability.

Corollary. 9.3.5 in [7] *Assume that $f\in L_2(\mathbb{R}^d)$ is compactly supported. Then the following are equivalent:*
(i) $\{f(\cdot-k)\}_{k\in\mathbb{Z}^d}$ is a Riesz sequence.
(ii) For every $\xi\in\mathbb{R}^d$, there exists an $k\in\mathbb{Z}^d$ such that $\widehat{f}(\xi+2\pi k)\neq 0$.

7. Limitations

Although stability is one of the major topics for my project, the proofs, and text regarding stability were very involved and required background material from areas of math that I was not familiar with. Because of this, I struggled to get a good grasp of the underlying proofs and assumptions that were being made in any relevant text related to stability. Additionally, due to my lack of familiarity with these concepts, it made it difficult to identify the most relevant sources of information and material. Due to the limited time available, I decided to spend most of it learning the background material regarding Distribution Theory and Fourier analysis to get a good grasp of these concepts and present results relating to Linear Independence.

8. Applications

Linearly independent integer shifts of a function are important for interpolation because they form a set of basis functions that can be used to represent any function that is band-limited to the same cut-off frequency.

Linear Independence is fundamental in the study of Shift Invariant Subspaces and Refinable Vector Functions

Linearly independent and stability of integer shifts are important concepts in the numerical solution of partial differential equations. These concepts are closely related to the concept of numerical stability, which is essential for obtaining accurate and reliable solutions to partial differential equations. For example, if a solution is stable under integer shifts, then it can be discretized effectively on a grid with a finite number of points.

Stability is widely used in approximation theory and wavelet analysis. One of the key uses of Stability is in the reconstruction of signals known as the Nyquist-Shannon Sampling Theorem [2]. The stability of integer shifts of functions allows us to reason about the signal's behavior under time translations. It ensures that the sampling process preserves the temporal structure of the signal, allowing us to reconstruct the original signal from the samples. Without the stability condition, the reconstructed signal might be distorted or incorrect.

9. Discussion

In this thesis, in section 3 we start by introducing and discussing various types of Distributions 3.1, 3.2, 3.5 and few of their properties and lastly discuss their relation to the Lebesgue Spaces 3.3, 3.5. Then in section 4 we discuss the Fourier Transform 4.1, Inverse Fourier Transfrom 4.2, Fourier Series 4.3 and some properties of the same 1, 4.1, 4.2. We later mention some well-known results like the Fourier Inversion 4.3, Poisson Summation Formula 4.4, PSF for distributions 4.5 and the Parseval's and Plancherel's Theorems 4.6. Then in sections 5 and 6 we finally discuss Linear Independence and Stability of Functions. We present the important results 1, 2, 5.1, 5.2 we proved in Linear Independence and later introduce and discuss some results in Stability 6.1, 1. Lastly, we talk about some limitations we encountered during the research process and applications.

Appendix A. Glossary

A.1. $\text{supp}(f)$: Support of a Function i.e $\text{cl}_X(\{x \in X : f(x) \neq 0\})$.

A.2. $\mathbb{T}^d$: $d-$ dimensional Torus i.e the cube $[0,1]^d$ with opposite sides identified.

A.3. $f(u) = \langle f, u \rangle$: Linear Functional i.e
for any a, b scalars , u, v vectors, $f(au + bv) = af(u) + bf(v)$.

A.4. $L_p(\cdot)$: Space of all f defined on $\mathbb{R}^d$ and $\mathbb{T}^d$, such that $\|f\|_{L_p(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |f(x)|^p dx\right)^{1/p}$ and $\|f\|_{L_p(\mathbb{T}^d)} := \left(\int_{\mathbb{T}^d} |f(x)|^p dx\right)^{1/p}$ are finite.

A.5. $F_{\tilde{n}}(t)$: The Fejér kernel i.e $F_{\tilde{n}}(t) = \sum_{|k| \leq \tilde{n}-1} \left(1 - \frac{|k|}{\tilde{n}}\right) e^{ikt}$

A.6. $\mathcal{O}(f(\cdot))$: Big-$\mathcal{O}$ i.e
$f(n) = \mathcal{O}(g(n))$ as $n \to \infty \iff \exists C > 0, \exists N \in \mathbb{N}$ such that $|f(n)| \leq C|g(n)|$ for all $n \geq N$.

Appendix B. Preliminary Theorems

Theorem B.1. (Hölder Inequality) Let $E \subseteq \mathbb{R}^d$ be a Lebesgue measurable set, $1 \leq p \leq \infty$ and $1/p + 1/p' = 1$,

$$\left|\int_E f(t)g(t)dt\right| \leq \left(\int_E |f(t)|^p dt\right)^{1/p} \left(\int_E |g(t)|^{p'} dt\right)^{1/p'}.$$

Theorem B.2. (Lebesgue Dominated Convergence Theorem) Let $E \subseteq \mathbb{R}^d$ be a Lebesgue measurable set and g be an integrable function and $\{f_n\}$ be a sequence of measurable functions such that $|f_n(t)| \leq g(t)$ a.e. and $f_n(t) \to f(t)$ a.e. $t \in E$. Then

$$\int_E f(t)dt = \int_E \lim_{n\to\infty} f_n(t)dt = \lim_{n\to\infty} \int_E f_n(t)dt$$

Theorem B.3. (Fubini's Theorem) Let E be a measurable set of $\mathbb{R}^d$ and F be a measurable set of $\mathbb{R}^d$. If $f(t,s) \in L_1(E \times F)$, then $f(\cdot, s) \in L_1(E)$ a.e. $s \in F$, $f(t, \cdot) \in L_1(F)$ a.e. $t \in E$, and

$$\int_{E\times F} f(t,s)dm(t,s) = \int_F \left\{\int_E f(t,s)dt\right\} ds = \int_E \left\{\int_F f(t,s)ds\right\} dt$$

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