Mathemania Contest
February 24, 2022

Before you Start...

- Please show **all** your work!. Answers without supporting work will not be given credit.
- **The questions are in increasing difficulty.**
- You have ≤ 3 hours to complete.
- The contest is closed book, and the premise is that of an honor code. Kindly do not discuss solutions with your peers.
- **Tied scores will depend on who submits first.**
- A max score of 67/60 is possible.
- After completion, submit your solutions to mathmss@ualberta.ca.
- Remember, *having fun* is the main goal of the exam. After you're done with each question, take some time to ponder the significance of your results. **Good luck!**
- Problems:
  - P1 : Calculus.
  - P2 : Probability.
  - P3 : Probability and Number Theory.
  - P4 : Group Theory and Music Theory.
  - P5 : Geometry and Calculus.
  - P6 : Linear Algebra, Calculus and Quantum Mechanics.
- Problem Contributors:
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  - P4, P5 : Davidson Noby.
  - P6 : Muhammad Abul Fazal.

Question 1: (Calculus) [10 points]

Evaluate[1]:

$$\lim_{x \to 0} \frac{(x+4)^{\frac{3}{2}} + e^x - 9}{x}$$

¶ *(Bonus)* [+5 points]: Solve the problem *without* L'Hôpitals rule

Hint:

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1 \qquad \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

Solution: There are two solutions:

Solution - 1: Let $f(x) = (x+4)^{\frac{3}{2}} + e^x - 9$. The limit can be written as

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

Which is the definition of $f'(0)$. Thus the answer is $\frac{3}{2}(0+4)^{\frac{1}{2}} + e^0 = 4$ $\square$

Solution - 2: Using the hints we get

$$\begin{aligned} L &= \lim_{x \to 0} \frac{(x+4)^{3/2} + e^x - 9}{x} \\ &= \lim_{x \to 0} \frac{(x+4)^{3/2} - 8}{x} + \frac{e^x - 1}{x} \\ &= \lim_{x \to 0} \frac{(x+4)^{3/2} - 8}{(x+4) - 4} + 1 \\ &= \lim_{t \to 4} \frac{t^{3/2} - 4^{3/2}}{t - 4} + 1 \text{ (by putting } t = x + 4 \text{)} \\ &= \frac{3}{2} \cdot 4^{1/2} + 1 = 4 \end{aligned}$$

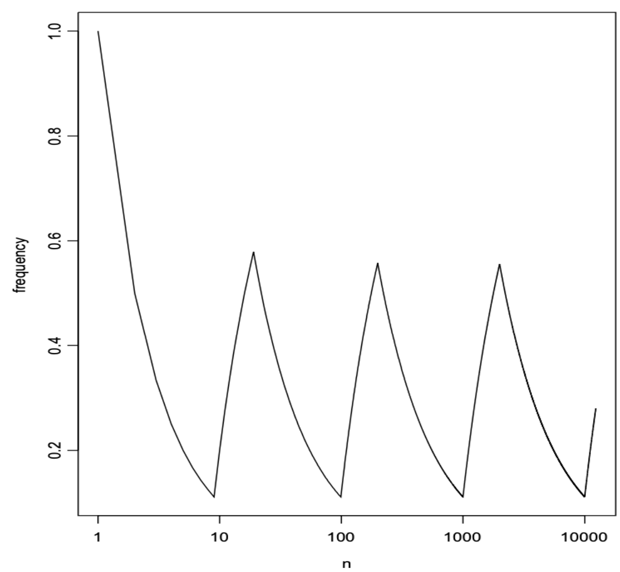
$\square$

[1] This Question can be solved in one line without without L'Hôpital's rule! :)

Question 2: (Probability) [10 points]

Say you have a set of numerical data, the probability that the *first significant digit* being a 1 or 2 or 8 is expected to be 11.1%. In fact this is not the case, in reality the number 1 appears 30.1% as the first significant digit, 5 appears 7.9% and so on 9 appears 4.6% as the first significant digit . This is due to **Benford's Law**. The definition of the first significant digit is the first digit that isn't '0'.An easy way

to understand Benford's Law is explained below-
Say you are in a raffle with two tickets labelled 1 and 2. The probability say $P(d = 1)$ (d is the first significant digit) of choosing the ticket labelled 1 is 50%. Suppose you add another ticket labelled 3 to the raffle now $P(d = 1) = 33.3\%$. Similarly you keep adding tickets labelled as successive numbers to the raffle and $P(d = 1)$ decreases until the number 10 is added and now $P(d = 1)$ is 20% ($\frac{2}{10}$). Now, $P(d = 1)$ increases till 19 and decreases from $20 - 99$. Repeatedly doing this we obtain the graph below



Benford took the average of the plot and found that $P(d = 1) = 30.1$

Benford's Law:

A set of numbers is said to satisfy Benford's law if the leading digit $d \in 1, \ldots, 9$ occurs with probability

$$P(d) = \log_{10}(d + 1) - \log_{10}(d) = \log_{10}\left(\frac{d + 1}{d}\right) = \log_{10}\left(1 + \frac{1}{d}\right)$$

Benford's Law above as stated is for decimal digits.

(a) Let us now consider binary digits (only for this part). Say we remove the number 0 from our data set. What is $P(d = 1)$? [4 points]

(b) Benford's law predicts a probability of $\log_{10}\left(1 + \frac{1}{d}\right)$ that a randomly chosen number in a given data set starts with the digit (or digits) d. If these probabilities are close to the actual probabilities for a given data set, then that data set is sometimes said to be Benford, e.g. "areas of countries are Benford".

Suppose a data set is Benford, can you say it's invariant under scaling (multiplication)? (For simplicity let's assume here for this question we scale by an amount of 2) [6 points]

Hint: The probability that $2n$ (n is a number from the data set) starts with d is equal to the probability that n starts with the digits $5d, 5d+1, 5d+2, 5d+3$, or $5d+4$.

Solution:
(a)

$$P(d=1) = \log_2\left(1+\frac{1}{1}\right) = \log_2(2) = 1 \text{ which is a } \textbf{100!} \text{ chance.}$$

The explanation for this question is largely referenced from - *sauce*.
(b) Following from the hint, this probability is

$$\log_{10}\left(1+\frac{1}{5d}\right) + \log_{10}\left(1+\frac{1}{5d+1}\right) + \cdots + \log_{10}\left(1+\frac{1}{5d+4}\right) = \log_{10}\left(\frac{5d+5}{5d}\right)$$
$$= \log_{10}\left(1+\frac{1}{d}\right)$$

Question 3: (Probability and Number Theory) [10 points]

The probability that two randomly chosen integers are co-prime is $\frac{6}{\pi^2} = \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{-1}$.

This question requires a lot of concepts from Number Theory so a "non-rigorous" observation based approach is mentioned below.

- Every integer has the probability "1" to be divisible by 1.
- A given integer is either even or odd hence has probability "1/2" to be divisible by 2.
- Similarly, an integer has a probability "1/3" to be divisible by 3. Because any interger is either the form $3k, 3k+1$ or $3k+2$

Conjecture: More generally, one integer chosen amongst "p" other integers has one chance to be divisible by p.

Complete this "proof".

Solution:

Two arbitrary integers a and b are coprime if and only if there is no prime p such that $p \mid x$ and $p \mid y$. These conditions are independent, so the probability a given p divides a and b is p^{-2}. These conditions are independent as p varies, so the probability a and b are coprime is

$$\prod_{p, p \text{ prime}} \left(1 - p^{-2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$$

The complete solution is referenced from *sauce*.

Question 4: (Group Theory ∩ Music Theory) [10 points]

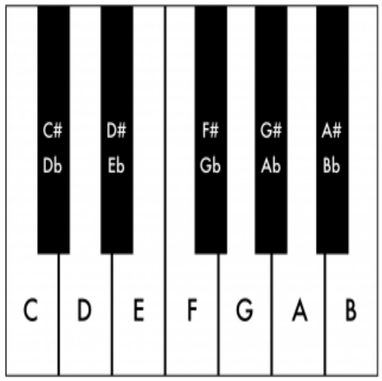


Figure 1: A snippet of the notes on a piano starting from C

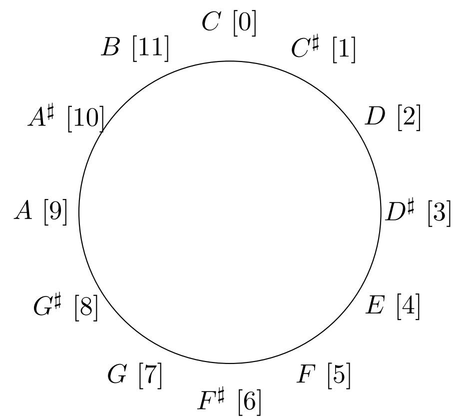


Figure 2: The Chromatic Scale illustrated in a circle

To make this question more interactive and fun we recommend to follow along via a virtual keyboard[2] but it is not necessary.

Music, as we know it is a universal language spanning vast cultures. Akin to how mathematicians want to the study the idea and behaviour of numbers by rigorously defining a system, musicians too have tried to come up with their own systems. Different cultures have come up with many different and (perfectly valid!) ways to understand music systematically. However, Music is inherently very complicated to study systematically. Consider a piece of music, perhaps it could be your favourite song, *Could you come up with a system to perfectly encapsulate everything? Would you really be able to convey what music means to others as such we do mathematics to others through symbols?*

One such attempt to understand music is through the lens of Western Music Theory. In this system, the notes that make up a song, say are part of the standard piano. Figure 1 shows a snippet of the piano that is divided into white and black notes starting at the note C. An observation one could make is that if you start at any note, and “move up" (play notes to the right), then you would reach the **exact same note** (i.e “the next" C) which sounds different yet incredibly familiar[3]. Thinking like a mathematician, you begin to sense a *cyclic* property. You gather all the notes starting at C to the last note B onto a circle as illustrated in Figure 1 forms what we call the **Chromatic Scale** as shown in Figure[4] 2. Then to commence the mathematics, you assign the notes to numbers and realize that you can describe each notes as integers in $\mathbb{Z}_{12}$ which is a collection of the *remainders* of any number divided with 12.

Recall

$$\mathbb{Z}_{12} = \{\mathbb{Z} \ni k : k \bmod 12 \in 0, 1, 2, 3, \ldots, 12\}$$

I.e

$$\mathbb{Z}_{12} = \{0, 1, 2, 3, 4, \ldots, 11\}$$

With the following association

$$\text{C} \longleftrightarrow 0,\ \text{C}^\sharp \longleftrightarrow 1,\ \text{D} \longleftrightarrow 2, \text{D}^\sharp \longleftrightarrow 3, \ldots, \text{B} \longleftrightarrow 11$$

It happens that this set forms *a group*[5] *under* addition[6]. With the cyclic nature as describe above, $\mathbb{Z}_{12}$ is rather aptly named the *cyclic group.*

[2] Hyperlink
[3] Try it in the virtual piano!
[4] Look! it's literally a clock
[5] Definition of a group would be given to those that don't remember
[6] here, meaning the only operation we perform is addition of two integers ‘normally’ and then taking its remainder

We call $x \in \mathbb{Z}_{12}$ a **generator** of a group if the set

$$\{x, x+x, x+x+x, \dots, 12x\} = \mathbb{Z}_{12}$$

where addition is after taking mod 12 and the equality means all elements of $\mathbb{Z}_{12}$ are present in the set. We denote this set in *the very same order* as

$$\langle x \rangle = \{x, \underbrace{x+x}_{2x}, \underbrace{x+x+x}_{3x}, \dots, \underbrace{12x}_{=0}\}$$

Clearly,

$$\langle 1 \rangle = \{1, 1+1, 1+1+1, \dots\} = \{1, 2, 3, \dots, 0\} = \mathbb{Z}_{12}$$

Which means that 1 is a generator of the group $\mathbb{Z}_{12}$ as it contains all the elments in $\mathbb{Z}_{12}$.

(a) (2 points) Show that if px is a generator of the group $\mathbb{Z}_{12}$, and $p \in \mathbb{Z}$ an integer then show that.

$$\langle p \cdot x \rangle = \mathbb{Z}_{12} \iff \gcd(p, 12) = 1$$

Conclude that there are exactly 4 generators for $\mathbb{Z}_{12}$
Hint:

You can set $x = 1$ for your convenience as it is a generator of the group.

Let G be any group under the operation (*), and $g \in G$ any element. Let p be a nonzero integer. We define g **applied to itself** p times as

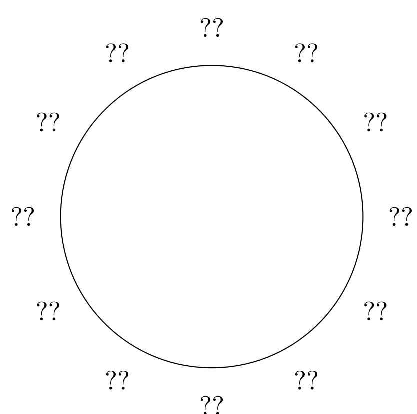
$$\underbrace{g * g * g * \dots g}_{p \text{ times}} = g^p$$

Use the fact that If g has finite **order $\boldsymbol{n}$**, then g^p has order $\frac{n}{\gcd(p,n)}$.

Order of an element of a group is the smallest positive integer n such that $g^n = e$, e being the identity element of the group. If such n doesn't exist we say the order is ∞. **Order** of a group is just the size of the group.

Solution: px generates a subgroup of order $12/\gcd(p, 12)$ by the hint, and this is the whole group $\mathbb{Z}_{12}$ (which has order 12) if and only if $12 = 12/\gcd(p, 12)$ or equivalently $\gcd(p, 12) = 1$.

(b) (2 points) It happens that 5 is a generator of $\mathbb{Z}_{12}$. Write down the elements in their order as per the definition of $\langle 5 \rangle$. Fill the corresponding musical notes in the following circle in the exact same order as in $\langle 5 \rangle$. Finish your solution by finding the other generators but you need not write the notes.



This is one of the central structures in music theory, so popular that it has a name, (aptly) called the **Circle of Fifths**[7].

Collection of notes stemming from these notes are widely used in both classical music and jazz, specifically in the form of **chords**[8].

¶ *(Bonus)* [+2 points]: Write out the elements that the other generator generates. (You need not write them as notes, the integers will suffice.)

Solution:

$$\langle 5 \rangle = \{5, 10, 3, 8, 1, 6, 11, 4, 9, 2, 7, 0\} = \mathbb{Z}_{12}$$

which is equivalent to $\{F, A^\sharp, D^\sharp, G^\sharp, C^\sharp, F^\sharp, B, E, A, D, G, C\}$

$$\langle 1 \rangle = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 0\} = \mathbb{Z}_{12}$$

which is equivalent to $\{C^\sharp, D, D^\sharp, E, F, F^\sharp, G, G^\sharp, A, A^\sharp, B, C\}$

$$\langle 7 \rangle = \{7, 2, 9, 4, 11, 6, 1, 8, 3, 10, 5, 0\} = \mathbb{Z}_{12}$$

which is equivalent to $\{G, D, A, E, B, F^\sharp, C^\sharp, G^\sharp, D^\sharp, A^\sharp, F, C\}$

$$\langle 11 \rangle = \{11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0\} = \mathbb{Z}_{12}$$

which is equivalent to $\{B, B^\sharp, A, A^\sharp, G, G^\sharp, F, E, E^\sharp, D, D^\sharp, C\}$

Note: *Earlier we mentioned that $\mathbb{Z}_{12}$ is called a cyclic group, and is it stands, a group is actually cyclic if and only if it can be generated by a single element.*

(c) (2 points) Prove inverse of a **generator** of $\mathbb{Z}_{12}$ (like the ones you showed above) is yet again, **another generator** itself.

Solution: $\mathbb{Z}_{12}$ is a cyclic group of order 12 and $\mathbb{Z}_{12} = \langle x \rangle$. When $p = 12$ by definition, if x is a generator then $\underbrace{x + x + x....}_{12x} = \langle px \rangle = \langle 12x \rangle = 0$. Also every two successive numbers are co-prime ie we have that $\gcd(p-1, p) = 1$ ie $\gcd(11, 12) = 1$. So it follows from part (a) results that, $\mathbb{Z}_{12} = \langle (p-1)x \rangle = \langle px - x \rangle = \langle -x \rangle$.

(d) (4 points) We mentioned **chords** earlier, and here we will talk about 2 different chords.

We call a **Major chord** with the **root** x defined as

$$x_M = \{x, x + 4, x + 7\} = [x, x + 4, x + 7]$$

An example is the C Major chord

$$C_M = 0_M = [0, 0 + 4, 0 + 7] = [0, 4, 7] = [C, E, G]$$

Go ahead and try playing these 3 notes simultaneously on the virtual keyboard, see how they sound like!

In a similar fashion, we call a **minor chord** with the **root** x to be defined as

$$x_m = \{x, x + 3, x + 7\} = [x, x + 3, x + 7]$$

Again, another example is the $G\sharp$ minor chord

$$G\sharp_m = 8_m = [8, 11, 15] = [8, 11, 3] = [G\sharp, B, D\sharp]$$

[7]By notation, this is a bit misleading as it's different because some of the notes with sharps ♯ are actually 'flats'; however the notes are still the same.

[8]Like Frank Sinatra's "Fly me to the moon"

Play them together and you get a 'sad'[9] tone.

Now it happens that when you play a *sequence* of chords with their root notes (say x, y) such that x, y are very close to each other on the circle of fifths[10], you hear something that sounds 'good'[11].

Write down a sequence of chords ≥ 5 chords with roots that are close to each other on the circle of fifths. Stick to either just a sequence of major chords or minor chords or mix them both if you are feeling adventurous.

Note: *Feel free to be as creative as possible since there are different progressions one could make. It is encouraged that you play the chords in the virtual piano too to get a feel for what you're creating.*

To end here, we have merely touched on the vast fields of both group theory and music theory that surprisingly, in this very special context, have *non-empty intersection.*

Solution: The following progression is a snippet of opening to "Frank Sinatra's Fly Me to the Moon" as referenced in an earlier footnote. Note, the chords are incomplete.

$$A_m, D_m, G_M, C_M; F_M, D_m, E_M, A_m, A_M$$

Notice how the root notes are exactly those on the circle adjacent to each other moving clockwise and anticlockwise.

[9] At least for now

[10] 'Very close' is purely subjective, but you can assume that $y = x \pm 1$ with $x, y \in \langle 5 \rangle$

[11] Why is this so?

Question 5: (Geometry & Calculus) [10 points]

Suppose you're a mathematician, going about your day working on unsolved problems to make a living. As it is with the nature of things, you hit a roadblock in your research. To take a break from it all, you decide to divert your attention to something else. Recently, you've become obsessed with triangles (How would I know?); and thinking of the internal angles of a triangle, you realized a stark paradox! To prepare (mathematically of course) for the shock, you define

$$\varepsilon(\Delta) = \theta_\Delta - \pi$$

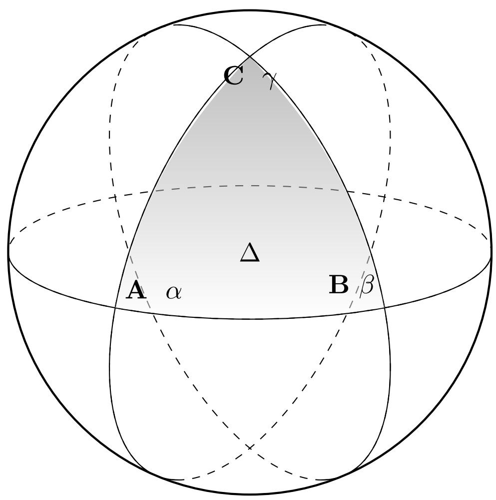
as the **angular excess** where θ_Δ is the sum total of the angle inside the triangle Δ.

You've known since high school that the sum of the angles in a triangle always add up to π. In this context, we have

$$\varepsilon(\Delta) = \pi - \pi = 0$$

and everything checks out.

However, you imagined yourself walking along the *great circles* of the Earth; circles whose centre coincides with the centre of the Earth and have radius exactly that of the Earth. Amongst all the possible great circles, you think of 3 specific great circles each perpendicular to each other and slicing the Earth into 8 *equal* parts as shown in the following figure.



You finally imagine yourself walking along these circles, starting at point **A** facing the line that leads to **B**, you make a left turn of $\frac{\pi}{2}$ and decide to walk in a *straight line* to point **C**. Then you turn by an angle of $\frac{\pi}{2}$ and walk in yet another straight line to point **B**. Finally, from point **B**, you make a right turn by an angle of $\frac{\pi}{2}$ onward to point **A**.

Then, the realization kicks in; you have walked across the boundary of a triangle with internal angles of $\alpha + \beta + \gamma = \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} > \pi$, or in other words

$$\varepsilon(\Delta) > 0$$

Thinking like a mathematician, you ask yourself *"Is this true in general?"* and decide to formulate a proof.

Hint: Breaking the fourth wall, you realized that to get the excess $\varepsilon(\Delta)$, you first need to find the area of the triangle $A(\Delta)$. To find the area of the triangle, you note that the total surface area of sphere is merely just the sum of the areas of *spherical lunes*, akin to surface area of slices of an orange. Spherical lunes are *defined* by the *"slice angle"*, i.e the angle that is at pointy ends of the slice.

(a) (6 points) Using Integration[12], derive an expression for the area of a spherical lune $\ell(\alpha)$ defined by an arbitrary angle α.

Solution: Exploiting spherical symmetry, let θ denote the elevation angle from the xy plane to the sphere, and R denote the fixed radius. Let φ denote the angle on the xy plane starting from the x axis. Then the surface area of a spherical lune of angle α is given by

$$\ell(\Delta) = \int_0^\alpha \int_{-\pi}^{\pi} \underbrace{(R\sin(\theta)d\theta)}_{\text{Arc length along circle } \perp xy \text{ plane}} \times \overbrace{Rd\varphi}^{\text{Arc length along } xy \text{ plane}} = 2R^2\alpha$$

(b) (2 points) Using your result from part (a), deduce an expression for the area $A(\Delta)$ of a triangle with internal angles α, β, γ bounded by 3 great circles (not necessarily perpendicular to each other as above).

Solution: We make the realization that the sum total area of the lunes (each lune has *also its diametrically opposite pair*) defined by the internal angles α, β, γ overshoots the total area of the surface of the sphere by exactly $2A(\Delta)$ (it double counts) on the triangle we're looking at as well as the triangle formed diametrically opposite. I.e we have

$$2\ell(\alpha) + 2\ell(\beta) + 2\ell(\gamma) - 2A(\Delta) - 2A(\Delta) = 4\pi R^2$$

$$\iff 4R^2(\alpha + \beta + \gamma - \pi) = 4A(\Delta)$$

$$\implies A(\Delta) = R^2(\alpha + \beta + \gamma - \pi)$$

(c) (1 point) Finally, piece your earlier solutions together to derive an analytical expression of $\varepsilon(R, A(\Delta))$ as a function of the radius R of the circle $A(\Delta)$.

Solution: Clearly, from above

$$A(\Delta) = R^2\varepsilon(\Delta)$$

(d) (1 point) Conclude that for any Δ with area $A(\Delta) > 0$ on a sphere bounded by *any* 3 great circles that

$$\varepsilon(\Delta) > 0$$

Solution:

$$\varepsilon(\Delta) = \frac{A(\Delta)}{R^2}$$

$$R > 0,\ A(\Delta) > 0 \implies > 0 \qquad \blacksquare$$

A natural question one might have is if it possible for for another geometry to exist where $\varepsilon(\Delta) < 0$ given that $\varepsilon(\Delta) > 0$ is nothing special. This is a question we want you to not answer here, but ponder.

[12] There is another more elementary argument to get to this result; that answer would be awarded 5/6 points.

Question 6: (Calculus, Linear Algebra and Quantum mechanics) [10 points]

Quantum Mechanics is the best model to understand the behaviour of tiniest things in our Universe. Atoms, electrons, etc. It encapsulates a strange reality of nature, that at the smallest scales objects behave both as particle (finitely spanning physical things) and waves (infinitely spanning modes of distribution of 'Something'(we just don't know what this 'Something' is, but there are different and equivalent interpretations of Quantum Mechanics which try to answer what's this 'Something', you can check out 'Copenhagenn', 'Many Worlds' interpretation etc)at the same time. The Language of one of the formulations of Quantum Mechanics is Linear Algebra but instead of having the comfort of $\mathbb{R}^n$ we have an infinite dimensional Vector Space of functions defined in Complex Numbers, these functions are called wavefunctions as they have all the information of the quantum particle (which also behaves as a wave) within them. Don't Get Scared Yet! This Problem is intended to give you a flavour of that, but is still solvable with pretty elementary math.

First Recall from Linear Algebra that given a matrix $A \in \mathbb{R}^{n\times n}$ and two vectors $x, y \in \mathbb{R}^n$, that $\langle x, Ay\rangle = \langle A^T y, x\rangle$ where $\langle a, b\rangle = a \cdot b$ is the dot product. Similarly we can define new "dot-product" (more precisiely an inner product) on a vector space of continuous functions. by $\langle f(x), g(x)\rangle = \int_a^b f(x)^* g(x)dx$ (where if $f(x) = u(x) + iv(x)$ then the complex conjugate that is $f(x)^* = u(x) - iv(x)$). Now, in the space of continuously differentiable functions, the derivative operator is a linear transformation (think why?, you don't need to show it here though).

(a) (2 points) Let $\psi : \mathbb{R} \to \mathbb{C}$ be a twice continuously differentiable function which satisfies the property that $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$ we call this a wave function. And this wavefunction acts as "vector" in our infinite dimensional space of continuously differentiable functions. Now we define an operator which we call Hamiltonian (physically it represents the sum of Kinetic and Potential Energy of a quantum particle) to be $\hat{H} = \frac{d^2}{dx^2} + V(x)$ where $V(x)$ is an arbitrary function of position (called Potential), so $\hat{H}\psi = \frac{d^2\psi(x)}{dx^2} + V(x)\psi(x)$. Show that $\hat{H}$ is a Linear transformation on the function space. (i.e. show that $\hat{H}(c_1\psi_1 + c_2\psi_2) = c_1\hat{H}\psi_1 + c_2\hat{H}\psi_2$, where c_1 and c_2 are constants).

Solution: Let ψ_1, ψ_2 be two wavefunctions and $c_1, c_2 \in \mathbb{R}$, then
$\hat{H}(c_1\psi_1 + c_2\psi_2) = \dfrac{d^2}{dx^2}(c_1\psi_1 + c_2\psi_2) + V(x)(c_1\psi_1 + c_2\psi_2)$
$= \dfrac{d^2(c_1\psi_1)}{dx^2} + c_1V(x)\psi_1 + \dfrac{d^2(c_1\psi_2)}{dx^2} + c_2V(x)\psi_2$
$= c_1\left(\dfrac{d^2\psi_1}{dx^2} + V(x)\psi_1\right) + c_2\left(\dfrac{d^2\psi_2}{dx^2} + V(x)\psi_2\right)$
$= c_1\hat{H}\psi_1 + c_2\hat{H}\psi_2$
by distributive property and Linearity of Differential Operator.

(b) (4 points) We call a Linear Transformation A "Hermitian" if $\langle x, Ay\rangle = \langle Ax, y\rangle$. Now coming back to our space of wavefunctions. Show that $\hat{H}$ is Hermitian with respect to the inner product defined in the first paragraph i.e. show that $\int_{-\infty}^{\infty} \psi^* \hat{H}\psi dx = \int_{-\infty}^{\infty} (\hat{H}\psi)^*\psi dx$.

Solution: $\int_{-\infty}^{\infty} \psi^* \hat{H}\psi dx = \int_{-\infty}^{\infty} \psi^* \left(\dfrac{d^2\psi}{dx^2} + V\psi\right) dx = \int_{-\infty}^{\infty} \left(\psi^* \dfrac{d^2\psi}{dx^2} + \psi^* V\psi\right)$
$= \int_{-\infty}^{\infty} \left(\psi^* \dfrac{d^2\psi}{dx^2}\right) dx + \int_{-\infty}^{\infty} \psi^* V\psi dx$
Now it's easy to observe that
$\int_{-\infty}^{\infty} \left(\psi^* \dfrac{d^2\psi}{dx^2}\right) dx = \int_{-\infty}^{\infty} \left(\dfrac{d}{dx}\left(\psi^* \dfrac{d\psi}{dx}\right) - \dfrac{d\psi}{dx}\dfrac{d\psi^*}{dx}\right) dx = \left(\psi^* \dfrac{d\psi}{dx}\right)\Bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \left(\dfrac{d\psi}{dx}\dfrac{d\psi^*}{dx}\right) dx$

Now from the propety of wavefunctions $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = \int_{-\infty}^{\infty} \psi^* \psi dx = 1$, we know the area under the curve of ψ, ψ^* is finite, hence as $x \longrightarrow 0 \implies \psi(x), \psi^*(x) \longrightarrow 0$ (in limit sense). Hence also $\frac{d\psi}{dx}, \frac{d\psi^*}{dx} \longrightarrow 0$, this would mean $\left(\psi^* \frac{d\psi}{dx}\right)\Big|_{-\infty}^{\infty} = 0$, Hence

$\int_{-\infty}^{\infty} \left(\psi^* \frac{d^2\psi}{dx^2}\right) dx = -\int_{-\infty}^{\infty} \left(\frac{d\psi}{dx}\frac{d\psi^*}{dx}\right) dx$

Now, using chain rule again,

$\int_{-\infty}^{\infty} \left(\frac{d\psi}{dx}\frac{d\psi^*}{dx}\right) dx = \int_{-\infty}^{\infty} \left(\frac{d}{dx}\left(\psi\frac{d\psi^*}{dx}\right) - \psi\frac{d^2\psi^*}{dx^2}\right) dx = \left(\psi\frac{d\psi^*}{dx}\right)\Big|_{-\infty}^{\infty} - \int_{\infty}^{\infty} \left(\psi\frac{d^2\psi^*}{dx^2}\right) dx$

Now by using the limit argument from above again we can conclude $\left(\psi\frac{d\psi^*}{dx}\right)\Big|_{-\infty}^{\infty} = 0$, so in total we get,

$\int_{-\infty}^{\infty} \left(\psi^* \frac{d^2\psi}{dx^2}\right) dx = \int_{\infty}^{\infty} \left(\psi\frac{d^2\psi^*}{dx^2}\right) dx$

So this implies,

$\int_{-\infty}^{\infty} \psi^* \hat{H}\psi dx = \int_{\infty}^{\infty} \left(\psi\frac{d^2\psi^*}{dx^2} + \psi V\psi^*\right) dx = \int_{\infty}^{\infty} \psi\left(\frac{d^2\psi^*}{dx^2} + V\psi^*\right) dx = \int_{-\infty}^{\infty} (\hat{H}\psi)^*\psi dx$

Hence Hermitian

(c) (4 points) Ultimately, interpreting $\hat{H}$ as linear transformations which is also Hermitian. We can talk about eigenvalues of this operator. Assume there exists two eigenvalues E_1, E_2 corresponding to two eigenvectors ψ_1, ψ_2 such that $\hat{H}\psi_1 = E_1\psi_1, \hat{H}\psi_2 = E_2\psi_2$ (these are energy of the independent Quantum states of a Given Potential and the corresponding wavefunctions are called eigenstates). Recall how in Linear Algebra if we want to show vectors to be orthogonal we show their dot product to be zero. We can show that these eigenstates are orthogonal (Hence all eigenstates together forms an orthogonal basis of this infinite dimensional Vector Space!) . Hence using your results from above show that $\int_{-\infty}^{\infty} \psi_1^*\psi_2 dx = 0$.

Solution: In previous part we figured that $\hat{H}$ is Hermitian, so $\langle\psi_1, \hat{H}\psi_2\rangle = \langle\hat{H}\psi_1, \psi_2\rangle$, this means,

$$\int_{-\infty}^{\infty} \psi_1^*\hat{H}\psi_2 dx = \int_{-\infty}^{\infty} \psi_2\hat{H}\psi_1^* dx$$

But we know that ψ_1, ψ_2 correspond to two seperate eigenfunctions hence with different eigenvalues E_1, E_2, such that $\hat{H}\psi_1 = E_1\psi_1, \hat{H}\psi_2 = E_2\psi_2$, this implies $\hat{H}\psi_1^* = E_1\psi_1^*$ (as reflection along real axis doesn't change sclaing),this means

$$\int_{-\infty}^{\infty} E_1\psi_1^*\psi_2 dx = \int_{\infty}^{\infty} E_2\psi_1^*\psi_2 dx$$
$$\implies (E_1 - E_2)\int_{-\infty}^{\infty} \psi_1^*\psi_2 dx = 0$$

But since $E_1 \neq E_2, E_1 - E_2 \neq 0$ so this would directly imply,

$$\int_{-\infty}^{\infty} \psi_1^*\psi_2 dx = 0$$