MOMENTS OF QUADRATIC DIRICHLET *L*-FUNCTIONS

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The aim of this summer project is to investigate the *r*-th moments of quadratic Dirichlet *L*-functions at the center of their critical strip, i.e. we want to investigate

$$M_r(D) := \sum_{0<d<D} L(\frac{1}{2}, \chi_d)^r.$$

A well-known conjecture from the early 2000s due to Conrey et. al. stemming from random matrix theory posits that there exists some polynomial Q of degree $r(r+1)/2$ such that

$$M_r(D) \sim DQ(\log D), \ \ D \to \infty.$$

Shortly after this conjecture was made, Goldfeld, Hoffstein, and Diaconu made a far more precise conjecture which, in particular, tells one that there will be lower order corrections to the above conjecture as soon as $r \geq 3$. In fact, very recently Diaconu has refined these conjectures even further for function fields and made certain unknown coefficients in these conjectures precise. When $r > 3$, the conjecture is totally open in the number field case, though there are some approaches and results known when one works over function fields and $r = 4$.

The more precise aim of this project is to find a new proof of the $r = 1, 2, 3$ case over function fields using the theory of Eisenstein series over metaplectic groups. The existing proofs proceed by a more ad-hoc approach, and the idea is that using the theory of Eisenstein series one can find a more unified approach. Moreover, such an approach should also work over number fields when $r = 1, 2, 3$, but this will involve a bit more technical work. More ambitiously, such an approach is also expected to generalize to higher r as well if one works with infinite-dimensional groups.

To begin the project, we will review the classic work of Hardy, Littelwood, and Ingham on moments of the Riemann zeta function. Then we will study the Eisenstein series on $SL(2, \mathbb{R})$ and describe its Fourier expansion. Following this, we will study the metaplectic covering group of $SL(2, \mathbb{R})$ and see how the Fourier expansion of Eisenstein series on this group looks like. This is classical work due to Shimura, Goldfeld, and Hoffstein and happens to be related to the $r = 2$ moment question, though this is somewhat accidental from our perspective. Instead, the groups which should lead to the $r = 1, 2, 3$ moment questions are $SL(3), SL(4)$, and $SO(6)$. Working over function fields and in these concrete settings, we would like to prove the recent conjectures of Diaconu et. al for the moments using metaplectic Eisenstein series on these groups.