Integration Bee 2020

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Abstract

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1 Problems

Problem - 1

$$\int_0^{\pi/4} (\tan x)^{\sec x} \left(\ln \left((\tan x)^{\sec x \tan x} \right) + \frac{\sec^3 x}{\tan x} \right) dx$$

Solution - 1
Note that

$$\frac{d}{dx}(\tan x)^{\sec x} = \left(\ln \left((\tan x)^{\sec x \tan x} \right) + \frac{\sec^3 x}{\tan x} \right)$$

Therefore

$$\int_0^{\pi/4} \frac{d}{dx}(\tan x)^{\sec x} dx$$

$$= (\tan x)^{\sec x} \Big|_0^{\frac{\pi}{4}}$$

$$= (1)^{\sqrt{2}} - 0$$

$$= 1$$

QED

Problem - 2

$$\int_0^1 (\log(x))^{2020} dx$$

Solution - 2.1

There are a couple of ways to solve this problem I will propose a few below.

Gamma Function

$$\int_0^1 (\log(x))^{2020} dx$$

Note that the Gamma function is given by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \ \Re(z) > 0$$

$$u = -\log x$$

$$du = \frac{-1}{x} dx$$

Therefore after the change of bounds and substituing in our integral it becomes

$$\int_0^\infty (-1)^{2020} (u)^{2020} e^{-u} du$$

Therefore this becomes basically the gamma function evaluated at 2021.

$$\Gamma(2021) = 2020!$$

QED

Solution - 2.2

Using integration by parts one can repeat the process

$$u = (\log x)^{2020} \text{ , } dv = dx$$

$$du = 2020\left(\frac{(\log x)^{2019}}{x}\right)dx \text{ , } v = x$$

$$= (\log x)^{2020} x\Big|_0^1 - 2020\int_0^1 (\log(x))^{2019}dx$$

$$= -2020\int_0^1 (\log(x))^{2019}dx$$

Repeating this process 2020 times we get

$$(-2020)(-2019)...... - 1 = 2020!$$

QED

Solution - 2.3

By direct use of the reduction formulae of

$$I_n = \int (\log(x))^n dx$$

$$= I_n = x(\log x)^n - nI_{n-1}$$

$$= I_{2020} = (\log x)^{2020} x\Big|_0^1 - 2020\int_0^1 (\log(x))^{2019}dx$$

from above

$$= 2020!$$

QED

Problem - 3

$$\int_0^{1/2} \frac{x^2}{(1-x^2)^{3/2}} dx = \frac{A\sqrt(B) - \pi}{C}$$

find ABC where B is square free

Solution - 3

I will propose the easiest method of solving this problem

$$x = \sin(u) \text{ , } dx = \cos(u)du$$

Changing the bounds after the u - substitution

$$\int_0^{1/2} \frac{x^2}{(1-x^2)^{3/2}} dx = \int_0^{\frac{\pi}{6}} (\tan u)^2 du$$

Using the Pythogorean identity

$$1 + (\tan u)^2 = (\sec u)^2$$

$$= \int_0^{\frac{\pi}{6}} (\sec u)^2 - 1du$$

$$= (\tan u - u)\Big|_0^{\frac{\pi}{6}}$$

$$= (\frac{\sqrt(3)}{3} - \frac{\pi}{6})$$

$$= (\frac{2\sqrt(3)}{6} - \frac{\pi}{6})$$

$$A = 2, B = 3, C = 6$$

$$ABC = 36$$

QED

Problem - 4

$$\int_0^\infty \ln(1+e^{-x})dx = \frac{\pi^a}{b}$$

Find a + b

Solution - 4.1 Note that using the Maclaurin expansion we get

$$\int_0^\infty \ln(1+e^{-x})\,dx = \int_0^\infty \sum_{n=1}^\infty (-1)^{n-1}\frac{e^{-nx}}{n}\,dx$$

Now as Fubini's Theorem applies here we can swap the integration and summation to obtain ie:
The interchange of series and integral are justified for the first integral since

$$\sum_{n=1}^\infty \int_0^\infty \left|\frac{(-1)^{n-1}e^{-nx}}{n}\right|\,dx = \sum_{n=1}^\infty \int_0^\infty \frac{e^{-nx}}{n}\,dx = \sum_{n=1}^\infty \frac{1}{n^2} < \infty,$$

Therefore

$$= \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n}\int_0^\infty e^{-nx}\,dx$$

$$= \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n}\cdot\frac{1}{n}$$

$$= \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^2}$$

Now note that this sum is equal to

$$= \frac{1}{1} - \frac{1}{2^2} + \frac{1}{3^2} - .....$$

$$= \sum_{n=1}^\infty \frac{1}{(2n-1)^2} - \sum_{n=1}^\infty \frac{1}{(2n)^2}$$

$$= \left(\frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} +\right)..... - 2\left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2}....\right)$$

$$= \sum_{n=1}^\infty \frac{1}{n^2} - 2\sum_{n=1}^\infty \frac{1}{(2n)^2}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$= \frac{1}{2} \cdot \zeta(2)$$

$$= \frac{\pi^2}{12}$$

$$A + B = 14$$

QED

Problem - 5

$$\int_0^\infty \arctan(\frac{1}{x^2})dx = \frac{\pi^c}{\sqrt(d)}$$

Find c+d

Note: This is a hard problem so bonus points will be given if one solves it

Solution - 5.1

I will propose a bunch of different methods on how to solve this problem and the one which I thought of while solving it

Going for the special Feynman's integration trick of differentiation under the integral sign

$$I(a) = \int_0^\infty \arctan\left(\frac{a}{x^2}\right) dx$$

$$= I'(a) = \int_0^\infty \frac{x^2}{(a^2 + x^4)} dx$$

Substituting

$$u = \frac{x^4}{a^2} \implies a^2 du = 4x^3 dx$$

Now some of you if may realize where I am heading to and if you figured out that it is the beta integral you are absolutely right and using the relation with the gamma function

$$B(m,n) = \int_0^\infty \frac{u^{m-1}}{(1+u)^{m+n}} du$$

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Therefore our integral after the substitution becomes

$$I'(a) = \frac{1}{4\sqrt{a}} \int_0^\infty \frac{u^{(\frac{-1}{4})}}{1+u} du$$

Thus now in relation with the beta integral this becomes

$$= \frac{1}{4\sqrt{a}} B(\frac{3}{4}, \frac{1}{4})$$

Note : One can also use the Legendre duplication which states

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$$

and calculate the integral but a simpler method which I found out was to use the euler's reflection formula that is the value of $\Gamma(\frac{3}{4}) \cdot \Gamma(\frac{1}{4})$

Now one may wonder where did I get this from (for those who are wondering)it is from the famous euler reflection formula which is

$$\Gamma(x) \cdot \Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

and here $x = \frac{1}{4}$

and therefore

$$= \frac{1}{4\sqrt{a}}B(\frac{3}{4}, \frac{1}{4}) = \frac{\Gamma(\frac{3}{4}) \cdot \Gamma(\frac{1}{4})}{4\sqrt{a} \cdot \Gamma(1)}$$

Upon simplyfing

$$= \frac{\frac{\pi}{\sin(\pi/4)}}{4\sqrt{a}}$$

$$= \frac{\sqrt{2}\pi}{4\sqrt{a}}$$

$$= \frac{\pi}{\sqrt{8a}}$$

Therefore finally

$$= I(a) = \int \frac{\pi}{\sqrt{8a}} da$$

$$= \frac{\pi\sqrt{a}}{\sqrt{2}} + C$$

$$I(0) = 0 + C = 0$$

Therefore the constant of integration is 0

and finally therefore

$$I(1) = \int_0^\infty \arctan\left(\frac{a}{x^2}\right) dx = \frac{\pi}{\sqrt{2}}$$

Therfore c + d = 3

QED.

Solution - 5.2

However when I was finding for another method to solve this I came across this theorem called Glasser's master theorem.

$$I = \int_0^\infty \arctan\left(\frac{1}{x^2}\right) dx$$

Integration by parts yields

$$I = x \arctan(\frac{1}{x^2})\Big|_0^\infty + \int_0^\infty \frac{2x^2}{1+x^4} dx$$

As the integral is even

$$= \int_{-\infty}^\infty \frac{x^2}{1+x^4} dx$$

Now this is a standard integral which can be solved using elementary means but however using the Glasser's Masters Theorem we can rewrite the integrand as

$$= \int_{-\infty}^\infty \frac{1}{x^2+x^{-2}} dx$$

$$= \int_{-\infty}^\infty \frac{1}{(x^2-x^{-2})^2+2} dx$$

Now by the Glasser's master theorem, this equals

$$= \int_{-\infty}^\infty \frac{1}{x^2+2} dx = \frac{\pi}{\sqrt{2}}$$

Therefore c + d = 3

QED.

Note: One can also use the Ramanujan's Master Theorem to solve this integral but that is left to the reader to figure out.